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AN ALGEBRA

FOR

HIGH SCHOOLS AND ACADEMIES

BY

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PREFACE

The demand for the latest and most improved books has led the author to prepare the present volume. It is the fruit of his years of experience as a teacher of algebra.

The book will be found well adapted for use in high schools, academies, and other preparatory institutions. The latest methods and newest processes are fully presented throughout; at the same time, all those older features of the science which are of permanent value have been purposely retained.

The plan of the book is such that it may be profitably used by students of different grades. The following is suggested as a good course for the more elementary classes: Fundamental Principles, — Notation, Addition, Subtraction, Multiplication, Division; Equations, — the first six principles in Special Theorems and their application to Factoring, Highest Common Divisor, Lowest Common Multiple, Equations and Fractions; closing with Equations of two and three unknown quantities.

The regular course for the more advanced high schools and academies should embrace the first seventeen chapters, while schools preparing students for universities having more advanced requirements will find in the additional chapters which the book contains all that may be needed.

The plan of the book is the *practical* based upon the *scientific.* Students preparing for the professions of civil, mechanical, electrical, naval, and mining engineering will find the book specially well suited to their needs. It begins with the most elementary mathematical forms and grows simply, yet logically, at every step. From the simplest

representation of numbers by letters, the student is, in the very beginning, unconsciously led up to the equation.

The subject of the Equation, the chief instrument of all mathematical investigation, is not confined to one section, but is judiciously distributed throughout all the pages of the book. The equation is among the very first subjects to greet the student, and it remains with him to the last.

The subject of Factoring, a special feature of the book, is first thoroughly treated by itself, and is then constantly used throughout the volume. It is based upon the chapter on Special Theorems and Symmetry, where the student is taught how expressions are built up, and is then made to see how they may again be separated into their parts. The principles of factoring are immediately applied to quadratic and higher equations, highest common divisor, lowest common multiple, and fractions. A ratio is treated as a fraction and a proportion as an equality of two or more fractions,— hence their treatment immediately after the subject of Fractions. A section on Variation is added for the benefit of students of physical science.

Involution, Evolution, Theory of Exponents, Radicals, and Imaginary and Complex Numbers, built up and dependent upon one another in the order named, are treated as constituting one great whole.

The subject of Quadratics is thoroughly treated in a modern way. In simultaneous quadratics, as in factoring, the student is shown the way of discovering the sure and best way of working any example with all "cut and try" methods eliminated. The final chapters are mainly for schools requiring a more advanced course in algebra, and may serve the purpose of a graduate course. The synopses for review and the sample test questions will prove matters of prime importance.

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ANN ARBOR, MICHIGAN,
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CHAPTER I

NOTATION, DEFINITIONS, AND LAWS

SECTION I

INTRODUCTION TO NOTATION

1. How letters are used to represent number.

1. If a stands for apple, what will stand for 5 apples?

2. If b stands for 2 boys, what will stand for 16 boys?

3. If y represents 3 yards, what will represent 18 yards?

4. If z represents 12 zebras, what will represent 6 zebras?

5. A man had \$5 and earned \$3. What *indicated* operation will represent what he then had?

6. A boy had \$$a$ and gained \$$b$. What *indicated* operation will represent what he then had?

7. A young man had \$20 and spent \$6 of it for tuition. What *indicated* operation will represent what he then had remaining?

8. In Ex. 7, if x stands for \$20 and y for \$6, what will stand for the remainder?

9. If x represents one number and y another, write: their sum; their difference; their product; their quotient.

10. The sum of two numbers being 8, what will represent the second, if x represents the first?

11. If c is greater than 7, how much greater is it?

12. If c is a whole number, what is the next one above it?

13. Write seven numbers in order of magnitude, so that 5 shall be the middle one. How do you count?

14. Write seven numbers in order of magnitude, so that n shall be the middle one.

15. Express the double of a; the triple of b; also the sum of these results.

16. A boy had b books and $\frac{1}{3}$ as many pencils. What represents the total number of both?

17. If the number of hours in a day is represented by x, what will express the number of minutes in two days?

18. If a cords of wood cost \$$b$, what will c cords cost?

19. A man bought 3 horses. He gave for the first twice as much as for the second, and for the third c times as much as for both the others. If x represents the price of the first, how much did he give for the third?

20. A man had a flock of m sheep. He lost $2n$ of them and raised $10a$, after which he sold the flock at \$$c$ a head, taking his payment in cloth at \$$b$ a yard. What will represent the number of yards of cloth received?

2. Notation involving the use of the sign of equality.

Write an expression for each of the following:

21. \$$a$ diminished by \$$b$ is equal to \$$c$.

22. The sum of a, b, and c is equal to 15 diminished by x.

23. With \$$s$ I paid for 16 quarts of berries at t¢ a quart, and received z¢ in change.

24. A man can travel a miles at the rate of b miles an hour in the same time that a boy can travel x miles at the rate of y miles an hour.

25. A father works p days at \$$q$ a day, and his son works r weeks at \$$s$ a week. With the money they buy t horses at \$$u$ apiece, and have \$$v$ remaining.

3. Letting x stand for one of the unknown numbers, find an expression for each of the following statements:

26. A number added to twice itself is equal to 12.

27. Three times Arthur's age increased by 5 equals 68.

28. The number 33 is divided into two such parts that the greater is 5 more than 6 times the less.

29. A merchant had two pieces of cloth. The longer piece lacked 10 yards of containing three times as much as the shorter. Both pieces together had 90 yards.

30. A man said of his age:

> "If to my age there added be
> Its half, its third, and three times three,
> Six score and ten the sum will be.
> So go away, don't bother me."

31. One number is 10 less than another. If 3 times the less be taken from 5 times the greater, the remainder will be 7 times the difference between the two numbers.

32. A number represented by x hundreds, y tens, and z units equals 216.

33. Five eggs at x¢ a dozen will pay for two lemons at b¢ a gross.

34. A certain number divided by P gives a quotient Q with a remainder of R.

35. The sum of four consecutive numbers is 14.

NUMERICAL VALUES

4. PROBLEM. **To find the value of an arithmetical expression involving the ordinary symbols of operation in arithmetic.**

Rule. *Perform operations indicated by the signs* $+$ *and* $-$, *also by the signs* $\times$ *and* $\div$ *in regular order; but the operations indicated by the signs* $\times$ *and* $\div$ *must be performed before those indicated by the signs* $+$ *and* $-$.

Model Solution

$$8+9-6\div 2\times 4+5\div 5-3+7=8+9-12+1-3+7=10.$$

EXAMPLES

1. $14+8-6+2-10-5.$
2. $6+8\div 2-3+10\times 2\div 4.$
3. $160\div(20\div 5)-(5+6-1)+5.$
4. $2\times 8\div 4.$
5. $10\times(6-3)\div(4+2)-(5-4).$
6. $5+15\times 3\div 9+6-3\times 3.$

Find the value of the following expressions when $a=4$, $b=2$, $c=3$, $d=1$:

7. $a+b.$
8. $a+b+c+d.$
9. $a-b.$
10. $a-b+c+d.$
11. $a-b+c-d.$
12. $a+b-c-d.$
13. $a\times b\times c.$
14. $a\times b\times c\times d.$
15. $a\div b\times c.$
16. $a\times b\div d\times c.$
17. $a\times c\div b\div d.$
18. $a\div b\times b\div d\times c.$
19. $6a\div b+3c\times d.$
20. $2a\div b+5c\div 3d+b.$
21. $a\div(c+d)+(a-d)\div c+b.$
22. $6a\div 2(c+d)+c-d.$
23. $(a+b)\times(b+c)\div(a+d).$
24. $5(a-b)\div 2(a+d)\times(6a\div b\div c\div d).$
25. $\dfrac{a+b}{a-b}.$
26. $\dfrac{a-b}{a+b}.$
27. $\dfrac{5a-4b}{c+3a}.$
28. $\dfrac{2a+b+c+d}{3a-2b+c-4d}.$
29. $\dfrac{a+2b+3c-5d}{5a-b-2c}.$
30. $\dfrac{4a-(c+d)}{a+b-c+3d}.$
31. $\dfrac{10a\div b+c-d}{2a\times b-c-d}.$
32. $\dfrac{(d+c)\div a+(c-d)\times b}{(a-b)\times 2c-(d+a)\div d+b}.$
33. $\dfrac{2a-b\times c\div 2d+a\times b-d}{3a\times b-c\times b\div c+a\div b-a\times c}.$

5. Solve the following problems, letting x stand for one of the unknown numbers in each :

1. A certain number added to itself equals 16. What is the number ?

2. What number added to the double of itself is equal to 45 ?

3. Mary and Laura together have 75 cents. Laura has 25 cents less than Mary. How many cents has each?

4. The difference between two numbers is 25; and if twice the less be taken from 3 times the greater, the remainder will be 80. What are the numbers?

5. The sum of two numbers is 180 and the difference 10. What are the numbers?

6. A man went to a bank with a check for \$36 and asked to have it cashed in half-dollars, quarters, dimes, and half-dimes, of each the same number. What was the number?

7. A man bought a horse and carriage for \$$x$. He tried to sell them at a profit of 10%, but failed of so doing by \$50. His loss on the investment was 15%. Find x.

8. A pupil was told to divide $\frac{1}{2}$ of a certain number by 4 and the other half by 2, and then take the sum of the quotients. He attempted to do this by dividing the whole number by 6, and testing this result with the one obtained by working according to the steps given him, found he was 5 out of the way. What was the original number?

9. A starts from a certain place and travels at the rate of 6 miles an hour. Ten hours afterward B starts from the same place at the rate of $7\frac{1}{2}$ miles an hour and attempts to overtake A, which he does in x hours. Find the value of x.

10. Divide \$5000 into two such parts that the first put at interest for 6 years at 10%, may equal the same amount as the second put at interest for 8 years at 5%.

11. A workman was hired for 60 days on condition that for every day he labored he should receive \$5, and for every day he was absent he should forfeit \$3. At the end of 60 days he received \$20. How many days did he labor?

12. What must be the value of x in order that $\frac{2m+x}{3x+69m}$ may be equal to $\frac{1}{24}$ when $m=\frac{1}{3}$?

13. A tourist has 5 hours to wait for his boat. How far may he ride in a coach which goes 6 miles an hour, so as to return in time, walking back at the rate of 4 miles an hour?

14. Potatoes are sold so as to gain 25% at 6 lb. for 10¢. What is the gain per cent when 5 lb. are sold for 12¢?

15. A man bought a motor cycle, and expected to sell it at 20% profit; but prices fell, and he was obliged to sell for $150 less than he expected. What did he pay for the cycle if he lost 10% on the cost?

16. There are three consecutive numbers. If $\frac{1}{2}$ of the first is added to $\frac{2}{3}$ of the second, and from this sum $\frac{1}{5}$ of the third is subtracted, the remainder will be 8. Find the numbers.

17. A number consists of 3 digits, of which the second is 2 greater than the first, and the third is equal to the sum of the first two. If the first and third change places, the number will be 297 more than the original number. Find the original number.

SECTION II

SYMBOLS OF NOTATION, AND DEFINITIONS

6. Quantity is the amount or the extent of that which may be measured. In algebra the word "quantity" is used synonymously with number, and, for convenience, is also applied to the symbols used to represent quantity.

ILLUSTRATIONS. 2, *a*, VI, etc., are called quantities, although they are but the symbols of quantities.

7. Number is quantity conceived as made up of parts, and answers the question "How many."

Thus,

$$3 = +1 + 1 + 1,$$

$$4 = +1 + 1 + 1 + 1,$$

$$3 + 4 = +1 + 1 + 1 + 1 + 1 + 1 + 1.$$

8. A **System of Notation** is a system of symbols. These symbols may be divided into six classes.

Symbols of
1. Quantity.
2. Operation.
3. Relation.
4. Aggregation.
5. Continuation.
6. Deduction.

A. SYMBOLS OF QUANTITY

9. Decimal, 0, 1, 2, 3, ⋯ 9, Arabic.

Literal
- $a, b, c, \cdots z$, common alphabet.
- $a', b'', c''', \cdots$, accented letters.
- a_1, b_2, c_3 (read a sub. 1, etc.), letters with subscripts.
- $\alpha, \beta, \gamma, \cdots \omega$, Greek alphabet.
- I, V, X, ⋯ M, Roman.

Special, ∞, infinity.

10. Two advantages of the literal over the decimal notation may be noticed. *First*, the symbols are more general in their meaning, and *second*, they may be traced throughout the solution of any problem. Hence the final result is often a general formula, which may be used in the solution of a special problem.

11. Laws of the Decimal Notation.

1. When numbers (the digits 0, 1, 2, ⋯ 9) are written side by side with no signs between them: (a) The one on the right is said to be in the first, or units order; the one next in the second, or tens order, etc., increasing to the left in a tenfold ratio. (b) The *sum* of the different orders is understood. Thus, $246 = 200 + 40 + 6$. (c) Each removal

of a number one place to the left or right multiplies or divides it by 10.

2. All numbers above 9 may be expressed by writing the different orders in succession, beginning with the highest.

3. The value of a number never changes.

12. Laws of the Literal Notation.

1. *Known Quantities* are those given in any problem, and are represented by letters taken from the first part of the alphabet, or by figures.

2. *Unknown Quantities* are those whose values are to be found by solving a problem, and are usually represented by letters taken from the latter part of the alphabet.

ILLUSTRATION. "If 2 lb. sugar cost 10¢, what will 3 lb. cost?" may be expressed in the literal notation as follows: "If a lb. sugar cost b¢, c lb. will cost x¢."

3. When letters are written side by side, as abc, their product is understood.

4. When letters are written with a number, as $56abc$, the product of the number and letters is understood.

B. SYMBOLS OF OPERATION

13. Addition.

1. $+$, the perpendicular cross, is read "plus," and signifies that the quantity before which it is written is to be added.

ILLUSTRATION. Thus, $a+b$, read "a plus b," signifies that the number represented by b is to be added to a.

2. Σ, the Greek letter sigma, placed before a term, is equivalent to the expression "the sum of such terms as." The term before which it is placed is a general type of those to be added together.

ILLUSTRATIONS.

$\Sigma a = a + b + c + \cdots + z.$

$\Sigma a = a + b + c + d$, when limited to four letters.

$\Sigma a^2 + \Sigma\, 2\,ab = a^2 + b^2 + 2\,ab$, when two letters are involved.

$\Sigma a^2 + \Sigma ab = a^2 + b^2 + c^2 + ab + ac + bc$, when three letters are involved.

14. Subtraction.

1. $-$, a short horizontal line, is read "minus," and signifies that the quantity before which it is placed is to be subtracted.

ILLUSTRATION. Thus, $a - b$, read "a minus b," signifies that b is to be subtracted from a.

2. $\sim$, an s-shaped symbol written horizontally, is read "the difference between," and signifies that the numerical difference of the two numbers between which it is placed is to be taken.

ILLUSTRATIONS. Thus, $2 \sim 5$, read "the difference between 2 and 5," equals 3. But $2 - 5 = -3$. If $a = 9$ and $b = 4$, $a \sim b = 9 - 4 = 5$; or, if $a = 4$ and $b = 9$, $a \sim b = 9 - 4 = 5$.

3. $\pm$, $\mp$, the plus and minus signs written one above the other, are read "plus and minus," or "minus and plus," and signify (1) the operations of addition and subtraction, as $a \pm b$, $a \mp b$; or (2) positive and negative, as ± 2, $\mp a$.

15. Multiplication.

1. $\times$, an oblique cross, is read "multiplied by" if read forward, or "times" if read backward along the line of reading, and signifies multiplication.

2. •, a dot, is read in the same way, and has the same signification as the last preceding symbol, $\times$.

3. The absence of a sign between factors indicates multiplication.

ILLUSTRATIONS. (*a*) \$$a \times b$ is read "*a* dollars multiplied by *b*," or "*b* times *a* dollars"; but never "*a* dollars times *b*."

(*b*) $3 \cdot 5 \cdot a \cdot b \cdot c$ is the same as $3 \times 5 \times a \times b \times c$.

(*c*) $10\,x^2yz$ signifies the product of 10, x^2, y, and z.

In this case the factors which go before are considered as multipliers of those which follow.

16. An *Exponent* is any symbol of number written to the right and a little above another symbol of number. It is often called an *Index*.

17. A *Positive Integral Exponent* shows how many times the other symbol of number is to be taken as a factor. It is another symbol of multiplication, in which all factors are equal.

ILLUSTRATIONS.

a^2 (read "*a* exponent 2," or "*a* second power") denotes $a \cdot a$.

a^4 (read "*a* exponent 4," or "*a* fourth power") denotes $a \cdot a \cdot a \cdot a$.

a^n (read "*a* exponent *n*," or "*a* *n*th power") denotes $a \cdot a \cdot a$, etc., to n factors.

The second power of a number is generally called the *square*, and the third power the *cube* of the number. Thus, a^2 and a^3 may be read "*a* square" and "*a* cube."

18. A *Power* of a number is the product arising from using it one or more times as a factor. It is a product where all the factors are equal.

Powers are *named* from the number of equal factors used.

Thus, a is called the *first* power of a,
aa is called the *second* power of a,
aaa is called the *third* power of a, etc.,

and may be expressed by the use of an exponent, as shown in the last article. a^1, however, is written a, the exponent *one* being not written, but understood.

NOTE. The meaning of fractional and negative exponents will be found in the chapter on The Theory of Exponents.

19. Division.

1. $\div$, a horizontal line between two dots, is read "divided by" if read forward, or "is contained into" if read backward, along the line of reading.

2. $:$, two dots, one above the other (a colon), is read the some way as (1), and, like it, signifies division.

3. $\frac{a}{b}$ and a/b, are the ordinary fractional forms in which division is indicated. Both are read "a divided by b."

4. $b\underline{)a}$, read "a divided by b," signifies short division.

5. $b\,)\,a\,(c$, or $a\underline{|b}$ over c, read "a divided by b equals c," signifies long division.

20. A *Root* of a number is one of the equal factors of the number.

Roots are *named* from the number of equal factors into which the power is to be resolved.

Thus, one of *two* equal factors is called the *second* root, one of *three* equal factors is called the *third* root, and so on.

The second root of a number is also called the *square* root, and the third root the *cube* root of the number.

21. The *Radical Sign*, $\sqrt{\ }$, is used to indicate the square root of a number. Any other root is indicated by a figure or letter placed in the opening of the sign.

Thus, $\sqrt[3]{8}$ signifies the third root of 8; $\sqrt[n]{a}$ signifies the nth root of a, n being a positive integer.

It is supposed that the sign $\sqrt{\ }$ was originally the initial letter of the word "radix," or root.

C. SYMBOLS OF RELATION

22. 1. $:$, the colon, is read "is to," and signifies geometrical ratio. Thus, $a:b$ is read "a is to b." It may be read "as a is to b," or "a divided by b." Geometrical ratio is often expressed in two other ways, (1) $a \div b$, (2) a/b.

2. $=$, two parallel horizontal lines, is read "equals," or "is equal to," and signifies equality.

3. $::$, the double colon, is read "as," or "so is," or "equals," and signifies equality of ratios.

Thus, $a:b::c:d$ is read "a is to b as c is to d." It may be written $\frac{a}{b}=\frac{c}{d}$ and read in the same way as before.

4. $\equiv$, three horizontal parallel lines, is the symbol of identity, and is read "is identical to."

Thus, $a \equiv a$; $a+b \equiv b+a$; $2 \times 5 \equiv 5 \times 2$.

5. $>$, a V placed on its side, is the symbol of *inequality*, the opening being toward the greater quantity. Thus, $a>b$ is read "a is greater than b," and $5<8$ is read "5 is less than 8."

The symbols of negation are made by drawing a line through the above symbols. Thus —

6. $\neq$, read "is not equal to," or "does not equal."

7. $\not>$, $\not<$, read "is not greater than," and "is not less than."

8. $\not\equiv$, read "is not identical to."

D. SYMBOLS OF AGGREGATION

23. 1. ( ), the parentheses, signify that whatever is written within is to be treated as one quantity.

2. { }, the braces, signify the same as ( ).

3. [ ], the brackets, signify the same as ( ).

4. ——, the vinculum, or tie, signifies that whatever is written below, or above, is to be treated as one quantity. Thus, $10-\overline{2+5}$ signifies that $2+5$ is one quantity, and is to be subtracted from 10. The result in this case is 3. Again, $\dfrac{a+b}{a-b}$ signifies that $a+b$ is to be considered as one quantity and divided by $a-b$, which is also to be considered as one quantity.

5. |, a vertical line, signifies that the aggregate of a column of numbers is to be treated as one quantity. Thus, $(10-6+2)x$ may be written $\begin{array}{r|l} 10 & x \\ -6 & \\ +2 & \end{array}$

E. SYMBOLS OF CONTINUATION

24. 1. • • •, a series of dots, signifies "and so on," or "etc."

2. - - -, a series of dashes, signifies "and so on," or "etc."

Thus, 0, 1, 2, 3, ⋯, is read "0, 1, 2, 3, and so on"; $a+b+c+\text{---}+z$ is read "a plus b plus c plus, and so forth to plus z."

F. SYMBOLS OF DEDUCTION

25. 1. $\because$ is read "since," or "because," and signifies *reasons for*.

2. $\therefore$ is read "hence," or "therefore," and signifies *conclusion*.

POSITIVE AND NEGATIVE QUANTITIES.

26. **Positive** and **Negative** are words primarily applied to two concrete quantities which are, by the conditions of a problem, naturally opposed to each other.

27. Two quantities of the same kind are opposed to each other if, when added, any amount of the one destroys an equal amount of the other. Of two such quantities, if either is called positive, the other is called negative.

ILLUSTRATIONS. Credits and debits are opposed to each other, for when added any amount of the one will destroy an equal amount of the other. Thus, \$5 in cash will destroy a debt of \$5. Either one may be called positive, and the other will then be called negative.

Let the students explain the illustrations which follow, and give others of their own:

1. Propelling a boat up a river.
2. Holding a pail of water with the hand.
3. A movement of 10 feet along a line, and back again.
4. A movement of the mercury above and below a fixed point on the thermometer scale.
5. Several quantities which, in a merchant's business, tend to one result, and several which tend to an opposite result.
6. Several quantities which tend to opposite results in a student's endeavors to obtain an education.

28. Positive and negative quantities are *distinguished* from each other by the use of the signs + and −. Thus, a force of 5 pounds acting in one direction is represented by +5 lb., while a force of 10 pounds acting in an opposite direction is

represented by -10 lb. Unfortunately, this makes *two uses* of the signs $+$ and $-$:

1. As signs of operations of addition and subtraction.

2. As signs of distinction between magnitudes of opposite natures. But since a positive quantity has an additive nature, it may be treated as a number to be added; and since a negative quantity has a subtractive nature, it may be treated as a number to be subtracted. If no sign is prefixed to a quantity, the sign $+$ is understood.

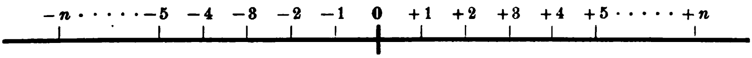
29. An **Arithmetical Number** is an abstract number and has properly no distinction as positive or negative, and is therefore represented by a figure without the sign $+$ or $-$.

30. The **Absolute Magnitude** of a number is the magnitude of the number considered independently of its sign $+$ or $-$.

Thus, a rise of 10° and a fall of 10° in heat are equal in *absolute* magnitude; likewise $+10$ and -10 are equal in *absolute* magnitude, whatever the unit may be.

31. An **Algebraic Number** is one that involves the additive or the subtractive idea. It is represented by a figure with the sign $+$ or $-$ prefixed, as $+2$, -3.

A letter represents both the *absolute magnitude* and the positive, or negative, idea of an algebraic number. Thus, x represents either a positive or a negative quantity. Its value may be $+2$ or -2 in any particular example. Algebraic numbers may be illustrated by the use of a scale.



If the divisions on the scale are taken as units of length, each number denotes not only distance, but also direction,

and answers the questions "*how far*" and "*in what direction* from zero, or the starting point."

This scale will be found useful in explaining an example in subtraction. Thus, from (-5) take $(+4)$. To go from $+4$ to -5, it is necessary to go to the left, hence the result is negative. It is also necessary to go 9 places, or units, hence the final result is -9.

32. The **Value of a Negative Quantity** is conceived to increase as its arithmetical value decreases.

ILLUSTRATION. $-3 > -5$, for a man who is in debt \$3, is better off than one who is in debt \$5, other things being equal.

33. One Quantity a is Greater than Another Quantity b when the algebraic difference $a-b$ is positive; and, in general, $-a < -b$, if $a > b$, a and b being positive integers.

Thus, $5 > 3$ if $5-3$ gives a positive result. $5-3=+2$. $\therefore\ 5 > 3$. $-3 > -5$ if $(-3)-(-5)$ gives a positive result. $(-3)-(-5)=-3+5=+2$. $\therefore\ -3 > -5$. According to the latter part of the definition, $-\$5 < -\$3$ if $5 > 3$, which is obviously true.

34. A **Quantity passing through Zero** changes its sign. This is assumed as a fundamental truth of the doctrine of positive and negative quantities.

ILLUSTRATION. Suppose the scale used in Art. **31** represents a thermometer scale. If the mercury stands at 4° below zero (marked $-4°$) at one hour, and at $-3°$ the next hour, the temperature is rising; and if it continues to rise, the mercury will come to be at 0°, passing which it will successively be at $+1°$, $+2°$, etc. It will be observed that the change in sign took place in passing through zero.

SECTION III

MATHEMATICAL FORMS OF EXPRESSION

35. A **Mathematical Expression** is any symbol, or combination of symbols, that represents number.

36. A **Term** is any expression which is governed throughout by the sign + or − prefixed. The parts of an expression which are connected by the signs + and − are called its terms.

ILLUSTRATIONS. $6a$, $2a^2bc^3$, $\frac{a}{b}$, $\frac{a}{a+b}$, $\frac{a-b}{a+b}$, $a \div b \times c \div d$, $(a+b) \div c$ are terms. The expression $2a^2 + \frac{2}{3}yz - \frac{x}{y} + \overline{a-b}$ is composed of four terms.

NOTE. It should be noticed that two or more expressions grouped by a symbol of aggregation constitute one term, as $\overline{a-b}$, $-2(a+b-c)$, for the first is governed by the sign + understood, and the second by the sign −.

37. A **Monomial** is an expression consisting of one term.

38. A **Binomial** is an expression consisting of two terms.

39. A **Trinomial** is an expression consisting of three terms.

40. A **Polynomial** is an expression consisting of more than one term.

The word "polynomial," however, is generally applied only to expressions consisting of more than three terms.

41. An **Integral Form of Expression** is one containing no fractions, as $a + b + c$.

42. A **Fractional Form of Expression** is one containing fractions only, as $a/b + c/d + 2/3$.

43. A **Mixed Form of Expression** is one containing both integral and fractional forms of expression, as $a+b/c+2/3+x-5$.

44. A **Rational Expression** is one free from fractional exponents and radical signs, or capable of being made free by performing the operations indicated, as 5, a, x, $\sqrt{4}$, $\sqrt[3]{8}$.

45. A **Surd** or **Irrational Expression** is one in which the root indicated by the fractional exponent or radical sign cannot be exactly extracted, as $x^{\frac{1}{2}}$, $\sqrt{3}$, $\sqrt[3]{x}$, $\sqrt{a-x}$.

46. The **Degree** of an algebraic expression is determined by the highest number of literal factors occurring in any one term, provided the expression is integral and rational.

ILLUSTRATIONS.

$3\,ax^2y$ is of the 4th degree. Why?

$2\,x^2-y$ is of the 2d degree. Why?

$6\,x^3+2\,xyz+3\,ab+10$ is of the 3d degree. Why?

The degree with respect to any *one letter* is determined in a similar way. The degree of $ab^2x^3-y^4$ with respect to x is the third; of b, the second; of y, the fourth.

47. **Homogeneous Terms** are those of the same degree.

48. The factors of any term may be divided into two groups: the **Base**, which serves to distinguish the term; and the **Coefficient**, or co-factor, which serves as a multiplier of the base.

ILLUSTRATION. In the monomial $6\,a^2b^3xy^4$, the numerical coefficient is 6, and $a^2b^3xy^4$ is its base. If y^4 is considered the base, $6\,a^2b^3x$ is the coefficient. If a^2b^3 is taken as the base, $6\,xy^4$ is the coefficient. Whatever factor, or combination of factors, is taken as the base, the remaining factor constitutes the coefficient.

49. Similar Terms are those having a common base.

ILLUSTRATIONS. $5\,ax$ and $6\,ax$ are similar, since they have the common base ax. $3\,ax$ and $7\,bx$ are similar with respect to the common base x.

50. Dissimilar Terms are those having no common base.

ILLUSTRATION. $3\,ay$ and $10\,xz$ are dissimilar, for they have no common base.

51. A more complete definition of a system of notation can now be understood, and may be stated as follows:

A **System of Notation** is a system of symbols by means of which, quantities, the operations to be performed upon them, and the relations existing between them, may be expressed more concisely than by the use of words.

52. Algebra is that branch of mathematics which treats of the science of universal numbers called quantities, and of their relations to one another; and also of the art of making mathematical investigations by the use of what are known as algebraic forms of expression, the chief one of which is the equation.

EXERCISE IN READING AND DESCRIPTION

Read and describe the following mathematical expressions according to Arts. **35–50**:

1. $3xz + \frac{2}{3}\,ab - \frac{x-7}{4}$.
2. $(a - b)\,x - \overline{2 + 4\,a}$.
3. $\frac{1}{2}\,ab - \frac{7}{5}\,axy + 8\,a^2b^3 + 3$.
4. $\frac{1}{2}\,(a - 3\,b) - 2\frac{1}{4}\{x - y\}$.
5. $2 : 3 :: 4 : x$.
6. $a'x - a_1y > a''z_3 - a_2$.
7. $(b - x)(\sqrt{b} + x) - \sqrt{b + x}$.
8. $\{a - (b - c)^2\}^3$.
9. $a - [b - \{c - (d - \overline{e - f} + g)\} - h]$.
10. $\sqrt[3]{\sqrt{a}} \div \sqrt[6]{a}$.
11. $\$\,a \times 4 \not\equiv \sqrt{64}$.
12. $(a + b + \cdots)^3 = \Sigma a^3 + \Sigma 3a^2b + \Sigma 6\,abc$.

SECTION IV

LOGICO-MATHEMATICAL TERMS

53. A **Proposition** (**Prop.**) is a statement of something to be considered or done.

ILLUSTRATIONS. It is snowing. Two plus three equals five. Find the value of x^0. In proving division, should the divisor be multiplied by the quotient, or the quotient by the divisor? Why?

54. An **Axiom** is a proposition which states a self-evident truth. It requires no proof.

ILLUSTRATION. The whole is equal to the sum of all its parts.

55. A **Theorem** is a proposition which states a truth to be proved by a course of reasoning.

ILLUSTRATIONS. Adding the same number to both terms of a fraction does not always increase the value of the fraction. $2 \times 6 \neq 17$.

56. A **Demonstration** (**Dem.**) is the course of reasoning by means of which the truth of a theorem is proved.

57. A **Corollary** (**Cor.**) is a consequent truth gained immediately from a preceding definition or demonstration. Usually it requires no proof.

58. **Analysis** is a statement in regular order of the different steps required in the solution of a problem.

59. A **Problem** (**Prob.**) is a question proposed for solution.

60. A **Rule** is a formal statement of the method of solving a general problem.

61. A **Solution** is (1) the answer to a problem; or, (2) the process of obtaining the answer to a problem.

SYNOPSIS FOR REVIEW, CHAPTER I

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     4. Literal Notation, advantages of, **10**.
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CHAPTER II

THE FUNDAMENTAL LAWS AND THEIR APPLICATION

SECTION I

ADDITION

62. Addition is the process of combining two or more quantities into the least number of terms consistent with the notation.

ILLUSTRATIONS. $a+a+a=3a$. $a+b$ added to $a+b$ may be written $a+b+a+b$. In this expression the process of addition is not completed, for the quantity is not yet expressed in the least number of terms consistent with the literal notation. $a+b+a+b=2a+2b$.

63. The **Sum** is the result in Addition.

LAWS FOR ADDITION

64. *A.* **Law of Order** (or Commutative Law).

The sum of two or more quantities is the same in whatever order they are taken.

ILLUSTRATION.

$a+b+c \equiv b+c+a \equiv c+a+b$, etc. For, if $a=2$, $b=3$, $c=4$, then

$2+3+4 \equiv 3+4+2 \equiv 4+2+3$, etc., $=9$.

Substituting numbers for letters is called *checking* one's work. It is a kind of proof. See check to Ex. 1, page 33.

65. *B*. **Law of Grouping** (or Associative Law).

Quantities to be added may be grouped in any manner.

ILLUSTRATION.

$a+b+c \equiv a+(b+c) \equiv (a+b)+c$. For, if $a=2$, $b=3$, $c=4$, then

$$2+3+4 \equiv 2+(3+4) \equiv (2+3)+4=9.$$

66. *C*. **Law of Distribution** (or Distributive Law).

Similar terms are united into one by adding their coefficients and affixing the common base to this result.

ILLUSTRATION.

$ax+bx+cx \equiv (a+b+c)x$. For, if $a=1$, $b=2$, $c=3$, $x=4$, then

$$4+8+12 \equiv (1+2+3)4=24.$$

NOTE. The Law of Distribution is really a law for multiplication, *i.e.* $(a+b+c)x \equiv ax+bx+cx$.

67. *D*. **Law of Signs.** In adding similar terms, if the terms are all positive, the sum is positive; if all negative, the sum is negative; if some are positive and some negative, the sum is positive if the positive quantities are in excess, or negative if the negative quantities are in excess.

ILLUSTRATIONS.
1. $+5a+(+7a)=+12a.$
2. $-5a+(-7a)=-12a.$
3. $+5a+(-7a)=-\ 2a.$
4. $-5a+(+7a)=+\ 2a.$

68. COR. 1. The sum of two quantities, the one positive and the other negative, is the numerical difference, with the sign of the greater prefixed. Give illustrations.

69. COR. 2. Addition in mathematics does not always imply an increase. If two quantities tend to the same end (positive or negative), the sum is an increase in that direction. If they tend to opposite ends, the sum is a decrease of the greater by the less. Give illustrations.

70. COR. 3. Adding a negative is the same as subtracting a numerically equal positive quantity. Give illustrations.

71. COR. 4. The sum of several quantities with opposite signs is equal to the sum of (1) the positive quantities; (2) the negative quantities; (3) the sum of these results.

ILLUSTRATION. The sum of $+7$, -9, $+3$, -8, and $+2$ is

$$\begin{aligned}(+7)+(-9)+(+3)+(-8)+(+2)&\\ &\equiv(+7)+(+3)+(+2)+(-9)+(-8) \text{ by law A}\\ &\equiv(+12)+(-17) \text{ by law B}\\ &\equiv-5 \text{ by Cor. 1.}\end{aligned}$$

72. PROBLEM 1. **To add monomials.**

Rule. — 1. *If the terms are similar, add their coefficients, and affix to this result the common base.*

2. *If the terms are dissimilar, write them in succession with their signs unchanged.*

Dem. Part 1 of the rule follows from law C. Part 2 is true because the whole is equal to the sum of all its parts; the sign before each term denotes its nature, *i.e.*, additive or subtractive.

Model Solution No. 1

Add $13\,m^2n,\ -10\,m^2n,\ -6\,m^2n,\ 5\,m^2n$, and $-4\,m^2n$.

1. $(13)m^2n+(-10)m^2n+(-6)m^2n+(+5)m^2n+(-4)m^2n$, coefficients set off.
2. $(13-10-6+5-4)m^2n$, by law C.
3. $(13+5-10-6-4)m^2n$, by law A.
4. $(18-20)m^2n$, by law B.
5. $-2\,m^2n$, by law D.

Check. Let $m^2=4,\ n=3$. Then

$13\,m^2n-10\,m^2n-6\,m^2n+5\,m^2n-4\,m^2n=156-120-72+60-48=-24.$

But $-2\,m^2n=-24$. $\therefore$ the work is correct.

Model Solution No. 2

Add $5\,ax,\ -2\,ex,\ 2\,mx$.

1. $(5\,a)x+(-2\,e)x+(2\,m)x$, coefficients set off.
2. $(5\,a-2\,e+2\,m)x$, by law C.

Model Solution No. 3

Add $5(x^2-y^2),\ 2\,\overline{x^2-y^2},\ -3\{x^2-y^2\}$.

1. $(5)(x^2-y^2)+(2)(x^2-y^2)+(-3)(x^2-y^2)$, coefficients set off.
2. $(5+2-3)(x^2-y^2)$, by law C.
3. $4(x^2-y^2)$, by laws B and D.

Model Solution No. 4

Add $4\,ey^2,\ 3\,ab,\ -2\,xy,\ -mn$.

$4\,ey^2+3\,ab-2\,xy-mn$, the quantities set down with their signs unchanged.

MENTAL EXERCISES

1. $(2)+(-5)+(3)+(-7)$.
2. $(5)+(-7)+(-6)+(4)$.
3. $(\frac{3}{4})+(-\frac{1}{2})+(-1)+(+\frac{1}{4})$.
4. $(-\frac{3}{4})+(1)+(-\frac{3}{7})+(-2)$.
5. $(\frac{5}{6})+(-1)+(-1)+(-5)$.
6. $(\frac{2}{3})+(-\frac{3}{4})+(-\frac{2}{5})$.
7. $(-3\frac{1}{2})+(-\frac{5}{6})+(-10)$.
8. $(2\frac{1}{4})+(-5\frac{1}{4})+(-17\frac{5}{6})$.
9. $(16)+(-20\frac{1}{5})+(2\frac{5}{8})$.
10. $(17)-(-3\frac{5}{6})-(+\frac{2}{3})$.

EXAMPLES

Add:

1. $3\,ax$, $6\,ax$, $-ax$, $2\,ax$, $-7\,ax$, $5\,ax$, and check.
2. $2\,by^2$, $-6\,by^2$, $-by^2$, $8\,by^2$, $3\,by^2$, $-2\,by^2$, and check.
3. $5\,ax^2$, $-2\,ax^2$, $3\,ax^2$, $-9\,ax^2$, ax^2, and check.
4. $-6\,a^2$, $2\,a^2$, $-5\,a^2$, $4\,a^2$, $-3\,a^2$, a^2, and check.
5. $\frac{3}{2}\,am$, $\frac{5}{2}\,am$, $-3\,am$, am.
6. $3\,m^2$, $-\frac{3}{4}\,m^2$, m^2, $-4\,m^2$.
7. $9\,b$, $\frac{3}{4}\,b$, $-\frac{5}{8}\,b$, $-8\,b$, $-\frac{5}{4}\,b$.
8. ax^2, $-by^2$, $-2\,cz^2$, $4\,mv^2$.
9. $x+n$, $-29(x+n)$, $30(x+n)$, $81(x+n)$, and check.
10. $4\,xy$, $3\,axy$, $-10\,mxy$, cxy.
11. $3\,xy^2z^3$, $-5\,xy^2z^3$, $-10\,xy^2z^3$, xy^2z^3, $-xy^2x^3$.
12. $-6(a-b)$, $12(a-b)$, $2(a-b)$.
13. $(a+b-c)vx$, $(a-b+c)vx$, $(-a+b-c)vx$.
14. $\frac{1}{2}(a-3\,b)$, $-\frac{1}{4}(a-3\,b)$, $\frac{1}{8}(a-3\,b)$, $-\frac{1}{10}(a-3\,b)$, and check.
15. $(a+b-c)(x-1)$, $(a-b+c)(x-1)$, $(-2\,a+b+c)(x-1)$.
16. $(3\,a-3\,c)(x+y^2)$, $-a(x+y^2)$, $c(x+y^2)$, $(5\,a+5\,c)(x+y^2)$.
17. $+28°$, $+10$ degrees, -3 degrees, $+2°$, $-7°$.
18. $3\,mn$ rods, $2\,mn$ rods, $6\,mn$ rods, $-5\,mn$ rods, $-mn$ rods, $-4\,mn$ rods.
19. \$100 cd, $-$\$15 cd, \$8 cd, \$2 cd, $-$\$5 cd, $-$\$7 cd.

73. PROBLEM 2. **To add polynomials.**

Rule 1. *Write all the terms in succession with their own signs, and then unite all similar terms.*

Dem. See laws A, B, C, and D.

Rule 2. *Write the polynomials so that similar terms shall be in the same column. Combine each column into one term and write the result underneath with its own sign. The polynomial thus formed is the sum sought.*

Dem. See laws A, B, C, and D.

Model Solution No. 1

Add $3a + 5b - 7c$ and $4a - 8b - 3c$.

1. $3a + 5b - 7c + 4a - 8b - 3c$, by law B.
2. $3a + 4a + 5b - 8b - 7c - 3c$, by law A.
3. $7a - 3b - 10c$, by laws C, D.

Model Solution No. 2

Add $a + 2b - 3c - q - x$, $2a + 5c - m$, $-5b - 3q + z$, $-5a + 9b - 7c + 5q$, $3c - 2q$, $-9c + 2x + 3b$.

$$
\begin{array}{rrrrrrr}
a & +2b & -3c & & -q & -x & \\
2a & & +5c & -m & & & \\
& -5b & & & -3q & & +z \\
-5a & +9b & -7c & & +5q & & \\
& & 3c & & -2q & & \\
& 3b & -9c & & & +2x & \\
\hline
-2a & +9b & -11c & -m & -q & +x & +z
\end{array}
$$

Let the student show where the laws apply in this example.

EXAMPLES

Add:

1. $x + y$, $x - y$, $x^2 + x - y$, and check.
2. $6x + 5ay$, $2ay - 3x$, $x - 6ay$, $ay + 2x$, and check.
3. $3ay - 7$, $8 - ay$, $2ay - 9$, $-3ay - 11$, $10ay - 13$.
4. $7x - \frac{3}{2}ba$, $3ab - \frac{10}{3}x$, $-6x + \frac{3}{4}ab$, $\frac{9}{2}x - ab$, $2ab + 4x$.
5. $2b - \frac{6}{5}a^2$, $2a^2 - \frac{3}{2}b$, $-\frac{5}{3}a^2 - 8b$, $\frac{4}{5}a^2 - \frac{2}{3}b$, $9b - 3a^2$.
6. $\frac{3}{2}a^2b^2 - \frac{7}{3}ab^4 + \frac{5}{4}axy$, $-7a^2b^2 - axy - 2ab^4$, $\frac{1}{2}ab - \frac{7}{5}axy + 8a^2b^3$.

7. $13\,ax^2 - 14\,y^2 + \frac{3}{2}\,ac^3 - \frac{1}{3}\,mn^2 - \frac{3}{4}\,a^3c + 2\,m^2n, \quad 4\,ax^2 + 15\,y^2, \; -17\,ax^2 + 4\,y^2 + \frac{2}{5}\,ac^3 + \frac{3}{4}\,mn^2 - \frac{1}{5}\,a^3c, \; 10\,y^2.$

8. $2\,a^{3n} + 4\,b^{2n}x^m - c^{2m}x^{2n},\; 4\,a^{3n} - 6\,b^{2n}x^m + 2\,c^{2m}x^{2n},\; 2\,a^{3n} + 2\,b^{2n}x^m - 4\,c^{2m}x^{2n}.$

9. $8\,a^2x^{2n} - 3\,xy^m, \; -5\,xy^m + 5\,a^nx, \; 9\,xy^m - 5\,a^nx, \; 2\,a^2x^{2n} + xy^m, \; -3\,xy^m + 5\,a^nx.$

10. $.02\,bx - 1.2, \quad -.026\,x + 3\,x^2, \quad 5\,x^2 - 3\,ax, \quad 1.2 + x^2 + 6\,ax, \; .3 + 5\,x^2 - 7\,ax.$

11. $x^n + 2\,x^{n-1} - 3\,x^{n-2} + 4\,x^{n-3} - 5\,x^{n-4} + 6,\; 7\,x^n - 8\,x^{n-1} + 9\,x^{n-2} - 10\,x^{n-3} - 11.$

12. $5\,(a^2+x^2), \; -7\,(a^2+x^2), \; 3\,(a^2+x^2), \; -10\,(a^2+x^2), \; 16\,(a^2+x^2).$

13. $2\,(a^2 - x), \; -a\,(a^2 - x), \; -3\,(a^2 - x), \; b\,(a^2 - x), \; 3\,a\,(a^2 - x), \; -4\,b\,(a^2 - x).$

14. $ax + by,\; cx + dy,\; ex + fy,\; 3\,x - 5\,y,\; x + y$, and check.

15. $3\,(x+y), \; -7\,(x+y) + 10\,(x+y)^2, \; -5\,(x+y)^2 - 2\,(x+y)^3, \; -6\,(x+y)^3.$

16. $(n+1)\,z^2 + (m-1)\,x^2 - (5-n)\,y^2, \quad (5-m)\,x^2 - (n-6)\,y^2 + (2-n)\,z^2.$

17. $2\,x - ay + bz,\; ax - 2\,y + 3\,z,\; bx + cy - dz$, and check.

18. $3\,x^4 - 4\,x^3 + 5\,x^2 - 6\,x + 7, \quad 8\,x^4 + 9\,x^3 - 10\,x^2 + 11\,x - 12, \; -7\,x^3 + 3\,x^2 - 1.$

19. $x^3 - x^2 + x - 1,\; 3\,x^2 - x^4 + 2\,x^3 - 5,\; x^6 - 7\,x^5 + 15 - x^3,\; 7 - 8\,x + 5\,x^3.$

20. $15\,x^5 - 7\,xy + 8\,x^3y + 4\,y^2, \; -7\,y^2 - 4\,xy - 11\,x^3y + 5, \; -15\,x^5 - 1 + 3\,x.$

21. $6\,y^3 + 7\,x^3 - 5\,x^2y + 5\,xy^2, \quad 3\,xy^2 - 4\,x^2y - 3\,y^3 - 4\,x^3, \quad -xy + 10 - 3\,y^2.$

22. $2\,x - y + x + z - 5\,a + 2\,w, \quad 7\,z - 5\,y + 8\,x - 9\,y + 5\,w - P, \; 6\,x - 5\,y + 2\,Q + z - 7\,P - 40, \; x^2 - y^2 - 3\,y + 4\,z - 9\,x + 3\,P.$

23. $\frac{1}{2}\,x - \frac{1}{3}\,y + \frac{2}{4}\,z + 7\,a,\; \frac{3}{5}\,x + \frac{5}{6}\,y - \frac{3}{5}\,z - \frac{1}{2}\,a,\; \frac{2}{3}\,x + \frac{5}{7}\,y - z - \frac{1}{3}\,a.$

24. $\dfrac{x}{2} + \dfrac{2\,y}{3} - \dfrac{5\,z}{6} + \dfrac{3\,a}{2},\; -3\,x + \frac{1}{4}\,y - \frac{3}{10}\,z - .6\,a,\; \frac{6}{7}\,a - \frac{5}{6}\,x + 2\,y - \frac{5}{8}\,z.$

25. $2\frac{1}{2}x^2 - 7\frac{1}{5}s^3 + 3\frac{1}{4}t - \frac{3}{5}u^2$, $3\frac{1}{3}u^2 - 4\frac{2}{5}x^2 - 3s + 7\frac{1}{5}s^3$, $-\frac{5}{4}s^3 + 6\frac{1}{7}t$.

26. $\frac{4}{5}a - \frac{5}{6}b^2 + \frac{6}{7}c^3 - \frac{7}{8}d^4 + \frac{8}{9}e^5$, $2b^2 + 3d^4 - 4c^3 - 5a - 6e^5$, $2\frac{1}{2}d^4 - \frac{3}{4}a + \frac{3}{7}b^2 - \frac{4}{5}e^5 - \frac{7}{6}c^3$, $\frac{4}{3}e^5 + \frac{3}{4}d^4 - .75a + .5c^3 - .6\frac{2}{3}b^2$.

27. $\frac{2\frac{1}{3}}{5}f - \frac{5}{2\frac{1}{3}}g + \frac{2\frac{1}{3}}{3\frac{2}{3}}h - \frac{3\frac{2}{3}}{2\frac{1}{3}}i + \frac{\frac{3}{4}}{\frac{5}{6}}j - \frac{.125}{\frac{1}{8}}k + \frac{\frac{3}{8}}{.375}l - .333\frac{1}{3}m$, $-\frac{3\frac{2}{5}}{17}f + \frac{2\frac{1}{3}}{5}g - \frac{5}{6}h + .66\frac{2}{3}i - 17j - .125k - \frac{3}{8}l + .0\frac{5}{6}$.

SECTION II

SUBTRACTION

74. **Subtraction** is the process of taking one quantity from another. The result should always be in the fewest terms consistent with the notation.

75. The **Minuend** is the quantity from which another is to be subtracted.

76. The **Subtrahend** is the quantity to be subtracted.

77. The **Remainder, or Difference,** is the result obtained by subtracting.

78. The **Proof** for the result is: remainder + subtrahend = minuend; or, minuend − remainder = subtrahend.

LAWS FOR SUBTRACTION

79. *A.* **Law of Order.** Quantities to be subtracted may be taken in any order.

ILLUSTRATION. $a - b - c \equiv a - c - b$.

For if $a = 17$, $b = 2$, and $c = 3$,

then $$17 - 2 - 3 \equiv 17 - 3 - 2;$$
$$12 \equiv 12.$$

80. *B.* **Law of Grouping.** Subtracting several quantities in succession is the same as subtracting their sum.

ILLUSTRATION. $a - b - c \equiv a - (b + c)$.

For if $a = 17$, $b = 2$, and $c = 3$,

then $$17 - 2 - 3 \equiv 17 - (2 + 3);$$
$$12 \equiv 12.$$

81. *C.* **Law of Distribution.** Similar terms are united into one by subtracting the coefficient of the subtrahend from that of the minuend and affixing the common base to this result.

ILLUSTRATION. $ax - bx \equiv (a - b)x$.

For if $a = 5$, $b = 3$, and $x = 10$,

then $$50 - 30 \equiv (5 - 3)10;$$
$$20 \equiv 20.$$

82. *D.* **Law of Signs.** Subtracting any quantity (positive or negative) is the same as adding a numerically equal and opposite (negative or positive) quantity, the sign of the result being determined by the Law of Signs for addition. (See demonstration of Prob. 1, Art. **83**.)

83. PROBLEM 1. **To subtract one monomial from another.**

Rule. *Add the subtrahend, with its sign changed, to the minuend.*

Dem. Let a be any quantity, and b any positive quantity.

Then $$a \equiv a + b - b, \text{ for } +b - b \equiv 0.$$

Let the two possible cases which may arise be: (1) From a subtract $+b$. (2) From a subtract $-b$.

$$a \equiv a + b - b, \text{ the key.}$$

(1)	(2)
$\begin{array}{r} a \\ +b \\ \hline a-b \end{array}$	$\begin{array}{r} a \\ -b \\ \hline a+b \end{array}$

(1) Taking away $+b$ from the key leaves $a-b$.

$$\therefore\ a-(+b)=a-b.$$

(2) Taking away $-b$ from the key leaves $a+b$.

$$\therefore\ a-(-b)=a+b.$$

MENTAL EXERCISES

1. $5+7-14+3-1$.
2. $3-5+6-10+2$.
3. $-2+5-6-2+10$.
4. $-10-6+2-1+5$.
5. $-6-7+5+9-3$.
6. $-1+0-7-3+8$.
7. $-0+0-3+5-19$.
8. $(-27)+(-4)+(-3)+(5)$.
9. $(18-1)+(-18+1)+(-16)$.
10. $(-23-1+2)+(-2+5)$.
11. $(-\frac{1}{5})+(-\frac{3}{7})+\frac{5}{7}$.
12. $-\frac{2}{3}+\frac{3}{4}-\frac{1}{6}-\frac{1}{2}$.
13. $(-\frac{1}{2})+(-2\frac{1}{2})+(3-4\frac{1}{3})$.
14. $(2\frac{1}{2}-3)+(4-7)+\frac{1}{2}-\frac{1}{2}$.
15. $7\sim 3+1\sim 8+3\sim 7$.
16. From $-\frac{2}{3}$ take 1, then add 5.
17. From $+\frac{5}{7}$ take 1, then subtract 1.
18. From $-\frac{7}{9}$ take $-\frac{3}{2}$, then subtract 1.
19. From $-\frac{3}{5}$ take $\frac{5}{7}$, then add 3.
20. $(1-3)-(-1-5)+(-\frac{2}{3}-1)-1\frac{1}{3}-\frac{2}{3}$.
21. $-(-(-1))-(-(+1))$.
22. $-3\frac{4}{5}-(+7\frac{4}{11})+(-\frac{5}{7}-(+1))$.
23. From $\frac{3}{4}$ minus 1 subtract 1 three times in succession.
24. From $\frac{3}{2}$ minus 1 subtract 1 four times in succession.

84. PROBLEM 2. **To subtract a polynomial from a monomial or a polynomial.**

Rule 1. *Write in succession the terms of the minuend and subtrahend, with the signs of the latter changed, and then combine all similar terms.*

Dem. See laws D, A, B, C.

MODEL SOLUTION No. 1

From $6a^2 - \frac{2}{3}bc + 3$ take $2a^2 + \frac{1}{2}bc - \frac{1}{4} + x$.

1. $6a^2 - \frac{2}{3}bc + 3 - 2a^2 - \frac{1}{2}bc + \frac{1}{4} - x$, by law D.
2. $6a^2 - 2a^2 - \frac{2}{3}bc - \frac{1}{2}bc + 3 + \frac{1}{4} - x$, by law A.
3. $4a^2 - \frac{7}{6}bc + \frac{13}{4} - x$, by laws B, C, D.

Rule 2. *Write the expressions so that similar terms shall be in the same column. Conceive the sign of each term of the subtrahend to be changed, and add the result to the corresponding term in the minuend.*

Dem. See laws A, B, C, D.

MODEL SOLUTION No. 2

From $x + 1 - y + 3z$ take $7q - 3y + 5z + \frac{1}{3}$.

$$\begin{array}{rl}
x + 1 - y + 3z \phantom{{}+7q} & \text{minuend,} \\
\frac{1}{3} - 3y + 5z + 7q & \text{subtrahend,} \\
\hline
x + \frac{2}{3} + 2y - 2z - 7q & \text{remainder.}
\end{array}$$

Let the student apply the laws used.

MODEL SOLUTION No. 3

$$\begin{array}{rl}
ax^2 + (b - c)y^s + z^{n-1} - 3abc + (5 - 7n)v & \text{minuend,} \\
bx^2 \qquad + cy^s + az^{n-1} - 16c - (2 + 3n)v & \text{subtrahend,} \\
\hline
(a - b)x^2 + (b - 2c)y^s + (1 - a)z^{n-1} + (16 - 3ab)c + (7 - 4n)v & \text{remainder.}
\end{array}$$

Let the student apply the laws used.

85. COR. 1. When a symbol of aggregation, preceded by the sign $-$, is removed from any expression it incloses, the sign of each term of that expression is changed.

Dem. This is evident, since the sign $-$ written before the symbol of aggregation signifies that the expression within is a subtrahend.

ILLUSTRATION.

$$a+(b-c)-\{d-e\}-[-f+g]-\overline{-h-i}-(m)-(-n)$$
$$=a+b-c-d+e+f-g+h+i-m+n.$$

Why not change the signs of the terms of the first binomial? What becomes of the sign $-$ before the symbols of aggregation? For what does it stand?

86. COR. 2. Any number of terms of a polynomial may be inclosed within a symbol of aggregation preceded by the sign $-$ by changing the signs of all the terms inclosed.

Dem. For, by removing the symbol according to Cor. 1, the expression reverts to its original form.

ILLUSTRATION. Introducing the second and third terms into parentheses, the fifth, sixth, and seventh terms into braces, the expression

$$a+2b-3c+4d-5e-6f+7g+8h$$
$$\equiv a+(2b-3c)+4d-\{5e+6f-7g\}+8h.$$

87. COR. 3. When one symbol of aggregation incloses one or more other symbols of like nature, begin their removal with either (1) the outside, or (2) the inside one.

ILLUSTRATION NO. 1. $2a-[b-\{2c-(3d-\overline{4d-5e})\}]$

$$=2a-b+2c-3d+4d-5e$$

at once, by beginning with outside one.

ILLUSTRATION NO. 2. $2a - [b - \{2c - (3d - \overline{4d - 5e})\}]$

$$
\begin{aligned}
&= 2a - [b - \{2c - (3d - 4d + 5e)\}] \\
&= 2a - [b - \{2c - 3d + 4d - 5e\}] \\
&= 2a - [b - 2c + 3d - 4d + 5e] \\
&= 2a - b + 2c - 3d + 4d - 5e.
\end{aligned}
$$

EXAMPLES

Subtract:

1. $2a^2 - 3mn + xy - 3$ from $3a^2 + 4mn - 2xy - 5$, and check.
2. $-7x - 2Q + 9s - 6R$ from $\frac{3}{4}x - \frac{2}{3}Q - .5s + R + t + u$.
3. $-\frac{2}{3}h - \frac{5}{6}k + 1.7l - 17m$ from $ah - \frac{2}{3}k + 2.7l - \frac{3}{2}m$.
4. $ax + by - cz - 714$ from $-dx - ey + fz - 215 + g$.
5. $a(x - y) - b(k - l) - c(m + n) + d$ from $e(x - y) + f(k - l) - (m + n)$.
6. $a + b - c - d + e - f + g - 3$ from $h - i + j + k - l - m + n$.
7. $5a^2 - 2ab + 7b^2 - 3ac - 6c^2$ from $-7a^2 - 5b^2 + 11c^2 + 5ba + 8ca$.
8. $x^2 + 2xy + y^2 - 3xz + 7z^2$ from $4z^2 + 5zx - 3xy - 5x^2 + 9y^2$.
9. $-15m^2 - 16n^2 + p^2 - 9mn$ from $pm - 2pn + 7nm - 17n^2$.
10. $ax + 2by - 3cz + 4mn$ from $3by + 3bx - 7nz + 5kn$.
11. $2a - 3b + 4c - 5d + 5$ from $-5a + 7d - m + 67 - x + y$.
12. $\frac{1}{2}a - \frac{3}{4}b + \frac{1}{4}c - \frac{1}{5}d - 7$ from $a + \frac{1}{4}b - \frac{1}{4}c + \frac{5}{6}d - 15$.
13. $2a' - 3c'd_2 - 6b^2c_3^3 + 19x_1^5y_2 + 3$ from $5a' + 4c'd_2 - 8b^2c_3^3 - 4x_1^5y_2 - \frac{1}{8}$.
14. $ax^2 + bxy + cy^2 + fx + gy + h$ from $a'x^2 - b'xy + c'y^2 + f'x + g'y - h'$.
15. $x_1x_2 + y_1y_2 + (x_1 + x_2)G + (y_1 + y_2)F + C$ from $x_1x_3 - y_1y_3 + G(x_1 + x_3) + F(y_1 + y_3) + C$.
16. $3x^4 - 5x^3 + 6x^2 - 7x + 8$ plus $9x^4 + 10x^3 - \frac{1}{2}x^2 + \frac{3}{4}x - \frac{7}{5}$ from $-2\frac{5}{8}x^4 + \frac{7}{5}x^3 - 2\frac{1}{5}x^2 - .375x + 12\frac{1}{2}$ plus $.1x^4 - .2x^3 + .3x^2 - .004x + .2\frac{3}{4}$.

17. $-5(m-n)+3(p+q)-\frac{7}{3}(r-s)$ from $-7(m-n)-a(p+q)-\frac{2}{3}(r-s)$.

18. $(a-b)x+(c+d)y-(e-f)z$ from $-bx-cy+fz$.

19. $gs-ht+iu-3$ from $(g-j)s-(k-h)t+(l-i)u$.

20. $(a+b)x-(c-d)y+(e+f)z$ from $x-y-z$, and check.

21. $-x+y-z$ from $(a-b)x-(c+d)y-(e+f)z$, and check.

22. $3x^n-5x^{n-1}+7x^{n-2}-11x^{n-3}$ from $\frac{2}{3}x^n-\frac{1}{2}x^{n-2}+\frac{2}{3}x^{n-3}-x^{n-4}$.

23. $x^5-x^4+x^3-x^2+x-1$ from $x^5+0x^4+0x^3+0x^2+0x+1$, and check.

24. $(x^2-x+\frac{2}{3})+(3x^2+5x-1)$ from $(x^2-x+\frac{2}{3})-(3x^2+5x-1)$.

25. $(?)-(3x^2-5x+\frac{1}{2})=-7x^2+2x-\frac{1}{3}$. Find (?), and check.

26. $(\frac{2}{3}x^2-7x-3)-(?)=\frac{2}{7}x^2-(a-5)x+3a$. Find (?).

27. $(5a^n-?)+(b^n-?)+(5a^{n-1}b^{n-1}-?)=2a^n-8a^{n-1}b^{n-1}+3b^n$. Find the value of (?) in each case.

Remove symbols of aggregation and collect terms:

1. $a-\{-b-c-(-b+c)-\overline{b+c}\}$. Check.
2. $2x-3-[x-\overline{y+z}+(5-x)-\{(x-y)-z\}]$. Check.
3. $-5-[-\{-(a-b)\}]+[-2\{-(a-y)\}]$. Check.
4. $ad-(a-2b)+\{d-2a+(c-3b)\}$. Check.
5. $b-c-\overline{(11b+c)-(6a-9b)}+(3b-5c)$. Check.
6. $2x+[10y+\{x+(10z+\overline{4y+2z}-8x)\}]$.
7. $(xy^2z+6z^2)-8xy^2z+(r-s)+\{-(8xy^2z-z)-(-12z^2+r)\}$.
8. $3b-[6c+9d-9b-(b+c)+\{6b-(c+d)\}]$.
9. $3a+(7x-2y)-\{a-(6x+3y)-(x-4y)\}$. Check.
10. $4a-\{2x-(4x-[6x-(8x-[10x-(12x-2x)])])\}$.
11. $(b+y)-(c-y)-(4b-6c)-(2b-\overline{2c-3y})$.
12. $\{4x^2-(9xy-y^2)\}-\{x^2-(16xy+y^2)\}+\{4y^2-(x^2-xy)\}$.

13. $3-\{3-(3-12\,a)\}+\{6\,a-(9-15\,a)\}+(-7\,a+21)$.

14. $[a-5\,b-\{a-(5\,c-\overline{2\,c-b}-4\,b)+2\,a-(a-\overline{2\,b+c})\}]$.

15. $-\{-[-(a^2-b^2)]\}+\{-[-(a^2+b^2)]\}$. Check.

16. $x-(y-z)-[-\{5\,x-y-z-(2\,x+3\,y+z)\}]$. Check.

17. $ax-[y+4\,x+\{3\,ax-(4\,y-3\,x)-5\,bc\}+6]$.

18. $-[3\,a-\{3\,b+c-(2\,a-\overline{c-b})+(b-a)\}]$.

19. $5\,x-3\,y-[3\,a+x-\{2\,y-a+(2\,x+y-4\,a)\}]$.

20. $1-[-1+\{-(1-\overline{x-y})-1\}-(x+y)]$.

21. $7\,m+\{3\,n-(4\,m+4)-(3+m-\overline{2\,n+3\,m})-5\}$.

22. $r-[3+7\,s+\{s-4\,r-(6+3\,s)\}]$.

23. $m-\{3\,x+2\,z-(4\,m-\overline{2\,x+z}-15)\}$. Check.

24. $4\,t-\{6\,w-z^2+(4\,w-\overline{t+3\,z^2}-w+3\,t)\}$.

25. $w-[+11\,v+5\,x^2-\{3\,v+7\,w+(9\,x^2-4\,v-w)\}]$.

Divide $a-b+c-d+e-f-g+h-i-j+k-l$ into

26. Binomials.

27. Trinomials.

28. Polynomials of four terms.

29. Polynomials of six terms.

Express in binomials, also in trinomials:

30. $-2\,a-3+7\,b-x+y-2\,z$.

31. $5-k+l-m-3\,n+px-qy-rz-a+b-c$.

32. $a-(2\,x-y)+(2\,ax-3\,h-4\,g)-9\,xy-5\,kx-47$.

33. $-(3\,x-8\,yz+7)-(f-g-s+t)-u-v+w$.

34. $r-[s-(t+\overline{u-v})-w]-x+(y-z)$.

If $a=12$, $b=10$, $x=5$, $y=8$, $z=-2$, find value of:

35. $\dfrac{(a^2-b^2)\cdot(x^2-z^2)}{20\,y-(4\,z)^2+yz}$.

36. $\left(\dfrac{a}{b}+\dfrac{x}{y}-\dfrac{y}{z}\right)\dfrac{b}{z}$.

37. $a^2x+ax^2-a^2y+ay^2+a^2z-az^2$.

38. $(a-b)^2+(x-y)^2-(a+z)^2$.

39. $(a-b)(x-y)-(b-z)^2\div xz$.

40. $\left(\dfrac{4}{a}-\dfrac{a}{4}\right)\dfrac{1}{y}-\left(\dfrac{b}{y}-\dfrac{y}{b}\right)\dfrac{1}{z}$.

41. $\dfrac{x^2+y^2+z^2}{a^2+b^2}-\dfrac{a^2-b^2}{y^2-(x^2+z^2)}$.

SECTION III

MULTIPLICATION

88. **Multiplication** is, primarily, the process of taking one number as many times as there are units in another.

The numbers to be multiplied together are called **Factors.**

89. The **Multiplicand** is the first factor, or the number to be multiplied.

90. The **Multiplier** is the second factor, or the number to multiply by.

91. The **Product** is the result in multiplication.

92. Since the above definition of multiplication does not include fractions and algebraic numbers as multipliers, a more general definition is necessary, and may be stated as follows:

Multiplication is the process of doing to one of two factors what is done to unity to produce the other.

ILLUSTRATIONS. Let it be required to find the product of $+6 \times +4$.

To obtain the multiplier $+4$ from unity, it is necessary to take $+1$ four times. $\because\ +4 = +1+1+1+1$.

To treat the multiplicand as unity was treated, it is necessary to take $+6$ four times, or $+6+6+6+6 = +24$.

1. $$\therefore\ +6 \times +4 = +24.$$

$+6 \times -4.$ $\quad -4 = -(+1)-(+1)-(+1)-(+1);$

2. $\therefore\ +6 \times -4 = -(+6)-(+6)-(+6)-(+6) = -24.$

$-6 \times +4.$ $\quad +4 = +1+1+1+1;$

3. $\therefore\ -6 \times +4 = +(-6)+(-6)+(-6)+(-6) = -24.$

$$-6 \times -4. \qquad -4 = -(+1)-(+1)-(+1)-(+1);$$

4. $\therefore\ -6 \times -4 = -(-6)-(-6)-(-6)-(-6) = +24.$

From (1), (2), (3), and (4), the Law of Signs for multiplication is derived by definition. See law D.

LAWS FOR MULTIPLICATION

93. *A.* **Law of Order.** The product of two or more factors is the same in whatever order they may be taken.

ILLUSTRATIONS. $ab \equiv ba$; $\$a \times b \equiv \$b \times a$; $2 \times 3 \equiv 3 \times 2$, $\because$ both $= 6$.

$\$a \times b \not\equiv b \times \$a$. Why? Check $ab \equiv ba$ when $a = \frac{3}{4}$, $b = \frac{2}{5}$.

94. *B.* **Law of Grouping.** The product of three or more factors is the same in whatever manner they may be grouped.

ILLUSTRATIONS. $abc \equiv (ab)c \equiv a(bc)$.

For, if $a = 2$, $b = 3$, $c = 4$,

then $$2 \cdot 3 \cdot 4 \equiv (2 \cdot 3) \cdot 4 \equiv 2(3 \cdot 4) \equiv 24.$$

95. *C.* **Law of Distribution.** The product of two algebraic expressions is equal to the sum of the products obtained by multiplying each term of the first by the second.

ILLUSTRATION. $(a + b + c)x \equiv ax + bx + cx.$

For, if $a = 2$, $b = 3$, $c = 4$, $x = 5$,

then $$(2 + 3 + 4)5 \equiv 10 + 15 + 20;$$
$$45 \equiv 45.$$

96. *D.* **Law of Signs.** When two factors have like signs their product is positive; when they have different signs their product is negative.

ILLUSTRATIONS.

$+a \cdot +b$ and $-a \cdot -b = +ab$, by definition.

$+a \cdot -b$ and $-a \cdot +b = -ab$, by definition.

97. *E.* **Law of Exponents.** The exponent of the product of two or more powers of the same quantity is equal to the sum of the exponents of the factors.

ILLUSTRATION. $a^m \cdot a^n = a^{m+n}$.

Dem. $a^m = a \cdot a \cdot a \cdot \cdots$ to m factors, by definition.

$a^n = a \cdot a \cdot a \cdot a \cdot \cdots$ to n factors, by definition.

$\therefore\ a^m \cdot a^n = a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot \cdots$ to $m+n$ factors

$= a^{m+n}$, by definition of exponent.

98. PROBLEM 1. **To multiply monomials.**

Rule. *Multiply together the numerical coefficients, observing the Law of Signs, and to this product affix the letters of all the factors, affecting each with an exponent equal to the sum of all the exponents of that letter in all the factors.*

Dem. See laws A, B, D, and E.

MODEL SOLUTION

Multiply together $3a^2b$, $-4ab^2c$, and $-7a^3b^3c^2d$.

$(3a^2b) \cdot (-4ab^2c) \cdot (-7a^3b^3c^2d)$

$= 3 \cdot -4 \cdot -7 \cdot a^2 \cdot a \cdot a^3 \cdot b \cdot b^2 \cdot b^3 \cdot c \cdot c^2 \cdot d$, by law A,

$= +84a^6b^6c^3d$, by laws B, D, and E.

MENTAL EXERCISES

Multiply together:

1. $-3a^2bc$, $2b^3c$, and $-5abc^2$. Check.
2. $7xyz$, $-4x^4y^3z^2$, $-xy^2z^3$, $-x^2y^4z^4$, and 3.
3. $6lm^2n^3x^4$, $8m^5n^6$, $-l^3x^5$, $-9l^4n^2x^3$, and $10a^2$.

4. $3\,a^5x^6,\ -5\,a^2,$ and $2\,a^7x^2.$

5. $3\,c^2y,\ 4\,cy^3,\ -acy,$ and $-2\,a.$

6. $4\,c^3y^2,\ 7\,acy^2,$ and $-3\,b^2y^3.$

7. $\frac{1}{2}\,abc,\ \frac{1}{3}\,ab,$ and $\frac{1}{4}\,bc.$ Check.

8. $\frac{2}{3}\,xy^2,\ -\frac{3}{2}\,x^2y,$ and $\frac{3}{4}\,xyz.$ Check.

9. $-\frac{5}{2}\,ax^2y,\ -\frac{7}{5}\,bmx,$ and $-\frac{2}{7}\,hf.$

10. $-\frac{2}{9}(yz)a,\ \frac{3}{2}\,b(xy),$ and $-\frac{5}{7}\left(\frac{xy}{2}\right).$

11. $-\frac{3}{7}\,x\left(\frac{a}{3}\right),\ -3\,x\left(\frac{b}{7}\right),$ and $\frac{7\,c}{3}.$ Check.

12. $mx^n,\ 2\,m^2x^3,\ -3\,ax^2,\ -5\,m^s,$ and $4\,a^3x^r.$

13. $-2\,c^md,\ -10\,ac^n,\ -d^2x,\ -4\,a^mx^n,$ and $-c^2.$

14. $-a^bc^d,\ 5\,e^fg^h,\ -\frac{2}{5}\,a^kc^ld^m,$ and $-\frac{1}{2}\,c^ne^pg^r.$

15. $\frac{5}{2}\,x^ay^bz^c,\ -\frac{2}{3}\,x^dy^ez^f,\ -\frac{3}{4}\,x^gy^h,$ and $\frac{5}{6}\,y^sz^t.$

16. $-5,\ a,\ \frac{2}{3},\ -\frac{5}{7}\,x^py^q,\ \frac{2}{5}\,x^n\left(\frac{y^t}{3}\right)\left(\frac{3\,z^s}{5}\right),$ and $-1.$

17. $\frac{7}{8},\ -\frac{4}{7}\,a,\ a^xb^yc^z,\ -b^sc^t,\ -2\,a^pb^b,$ and $0.$

99. PROBLEM 2. **To multiply a polynomial by a monomial.**

Rule. *Multiply each term of the polynomial by the monomial, and combine the partial products.*

Dem. See laws C and D.

MODEL SOLUTION

Multiply $5\,a^2kx^5-\frac{2}{3}\,bc^2h^3z-y^3z^2$ by $4\,a^2bc^3xyz.$

$$\begin{array}{l} 5\,a^2kx^5-\frac{2}{3}\,bc^2h^3z-y^3z^2 \\ 4\,a^2bc^3xyz \\ \hline 20\,a^4bc^3kx^6yz-\frac{8}{3}\,a^2b^2c^5h^3xyz^2-4\,a^2bc^3xy^4z^3 \end{array}$$

MENTAL EXERCISES

Multiply:

1. $3x^2 - 5y + z$ by 6, and check.
2. $6ab^2 - 3x^2y + 7$ by -8.
3. $7a + 5b$ by $-c$, and check.
4. $4b^2c + de - c$ by $3bc$.
5. $12ax^2 - x^3 + z^2$ by $-5a^2x^2$.
6. $3a^2 - 5x^3 - \frac{1}{2}y$ by $-8a^3xy^2$.
7. $-2x - 3y^2 - 4z^3$ by $-4x^4y^5z^6$.
8. $3x^2 - 5y^2 + 7z^3$ by $3xy^2z^3$.
9. $\frac{7}{5}x^3 - \frac{4}{7}y^3$ by $-\frac{4}{7}x^4y^5$, and state laws used.
10. $\frac{6}{7}x^2y^2 - \frac{2}{3}mn - \frac{3}{5}a^3b^3$ by $-\frac{7}{8}a^3b^2x$.
11. $\frac{1}{2}x^2 - \frac{2}{3}xy + \frac{3}{4}y^2 - \frac{4}{5}xyz$ by $-\frac{5}{12}xy^2z^3$.
12. $x^a - y^b + z^c$ by xyz, and state laws used.
13. $x^{a+m} - y^{b+n} + z^{c+p}$ by $x^ry^sz^t$.
14. $-\frac{2}{3}a^{x+1} - b^{y+2} - c^{z+3}$ by $a^{2x}b^{3y}c^{4z}$, and check.

100. PROBLEM 3. **To multiply a polynomial by a polynomial.**

Rule. *Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

Dem. This is the most general case of law C, *i.e.*, $(a + b + c)x = ax + bx + cx$. Since x may have any value, let $x = m + n$. Then

$$
\begin{aligned}
(a+b+c)(m+n) & \\
&= a(m+n) + b(m+n) + c(m+n) \\
&= (m+n)a + (m+n)b + (m+n)c, && \text{by law A} \\
&= ma + na + mb + nb + mc + nc, && \text{by law C} \\
&= am + bm + cm + an + bn + cn, && \text{by law A.}
\end{aligned}
$$

In a similar manner the rule may be proved for polynomials of any number of terms, so that in general,

$$(a+b+c+\cdots)\cdot(m+n+o+p+\cdots)=am+bm+cm+\cdots$$
$$+an+bn+cn+\cdots+ao+bo+co+\cdots+ap+bp+cp+\cdots.$$

A polynomial is said to be arranged with reference to a certain letter when the term containing the highest exponent of that letter is put first (in ascending series), or last (in descending series), the term containing the next higher exponent next, etc.

ILLUSTRATION. $5\,xa^4+x^5+10\,x^3a^2-5\,x^4a-a^5-10\,x^2a^3$ $=x^5-5\,x^4a+10\,x^3a^2-10\,x^2a^3+5\,xa^4-a^5$, arranged with reference to x, or $-a^5+5\,a^4x-10\,a^3x^2+10\,a^2x^3-5\,ax^4+x^5$, arranged with reference to a.

MODEL SOLUTION

Multiply $-5\,pm+3\,ab-7\,yx$ by $4\,xy-2\,ba-6\,mp$.

$$\begin{array}{lllllll}
3\,ab & -5\,mp & -7\,xy, & & & & \text{multiplicand arranged.}\\
-2\,ab & -6\,mp & +4\,xy, & & & & \text{multiplier arranged.}\\
\hline
-6\,a^2b^2 & +10\,abmp & +14\,abxy, & & & & \text{1st partial product.}\\
 & -18\,abmp & & +30\,m^2p^2 & +42\,mpxy, & & \text{2d partial product.}\\
 & & +12\,abxy & & -20\,mpxy & -28\,x^2y^2, & \text{3d par. prod.}\\
\hline
-6\,a^2b^2 & -8\,abmp & +26\,abxy & +30\,m^2p^2 & +22\,mpxy & -28\,x^2y^2, & \text{the product.}
\end{array}$$

EXAMPLES

Multiply:

1. $a+b$ by $a+b$, state laws used, and check.

2. $a-b$ by $a-b$, state laws used, and check.

3. $a+b$ by $a-b$, state laws used, and check.

4. x^2+xy+y^2 by $x-y$, state laws used, and check.

5. $x^2 - xy + y^2$ by $x + y$.

6. $10\,a^2x^2 - 7\,b^2y^2$ by $4\,a^2x^2 - 5\,b^2y^2$.

7. $x^2 - xy - xz + y^2 - yz + z^2$ by $x + y + z$.

8. $(x + y)(x^2 - xy + y^2)$ by $(x - y)(x^2 + xy + y^2)$.

9. $(x^2 - xy + y^2)(x^2 + xy + y^2)$ by $(x + y)(x - y)$.

10. $(x + y)(x^2 + xy + y^2)$ by $(x - y)(x^2 - xy + y^2)$.

11. $(x^4 - x^2y^2 + y^4)$ by $(x^2 + y^2)$.

12. $(x^{14} - x^7y^7 + y^{14})$ by $(x^7 + y^7)$.

13. $x^8 - x^6a^2 + x^4a^4 - x^2a^6 + a^8$ by $x^2 + a^2$.

14. $x^4 + x^2a^2 + a^4$ by $x^4 - x^2a^2 + a^4$.

15. $x^2 + 3\,x + 9$ by $x - 3$.

16. $(x^2 - 5\,x + 7)$ by $(x^2 - 5\,x + 7)$.

17. $(a + b + c)$ by $(a + b - c)$. Check.

18. $(ax + by - 1)$ by $(ax + by + 1)$. Check.

19. $(a^m + b^m)$ by $(a^n + b^n)$. Check.

20. $(2\,x^3 - 3\,xy + 3)$ by $(x^2 + 4\,xy - 2)$.

21. $6\,a^3 - 2\,b^2c + 3\,de^2$ by $a^2 - 5\,a^3b + 3\,e^2$.

22. $1 + y + y^2 + y^3 + y^4 + y^5$ by $1 - y$.

23. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$ by $x - y$.

24. $4\,a^5 + 3\,b^5$ by $4\,a^5 - 3\,b^5$.

25. $4\,abc + 3\,b^2$ by $4\,abc + 3\,b^2$.

26. $a - b + c + d$ by $a - b - c - d$.

27. $7\,a^2 + ay - 12\,y^2$ by $7\,a^2 - ay - 12\,y^2$.

28. $(1 + x + x^2)$ by $ax - 1 - x^2$.

29. $c^{16} - c^8x^8 + x^{16}$ by $c^8 + x^8$.

30. $81\,m^4 + 54\,m^3c + 36\,m^2c^2 + 24\,mc^3 + 16\,c^4$ by $3\,m - 2\,c$.

31. $(10 - 11\,x + 12\,x^2 + 13\,x^3 + 14\,x^4)$ by $(9\,x^2 - 10 - 11\,x)$.

32. $(a^5 - 9\,a^3 + 8\,a + 7 - 6\,a^4 - 5\,a^2)$ by $(4\,a^3 - 3\,a^2 + 2\,a - 20)$.

33. $(a-2d)(a-2d)(a-2d)$.

34. $x^{m+n}+y^{p+q}$ by $x^{r+s}-y^{t-u}$.

35. $\frac{3}{4}a^2-\frac{2}{3}b^2$ by $\frac{3}{4}a^2-\frac{2}{3}b^2$.

36. $\frac{1}{4}a^3+\frac{1}{3}b^3$ by $\frac{1}{4}a^3-\frac{1}{3}b^3$.

37. $x^{2m}+x^my^m+y^{2m}$ by x^m-y^m.

38. $3a^2-\frac{1}{2}b+c$ by $3a^2+\frac{1}{2}b-c$.

39. $\frac{2}{3}a-\frac{1}{2}b+\frac{1}{4}c$ by $\frac{1}{2}a+\frac{1}{3}b-\frac{1}{4}c$.

40. $(x-4)(x^2-6x+7)(x-5)(x+3)$.

41. $x^{2n+1}+x^{2n}y^{2n}+y^{2n-1}$ by $x^{2n-1}-y^{1-2n}$.

42. $(3x^4-5x^2+7x-3)(x-1)(x-2)$.

43. $(5x^5-4x^4+3x^3-15)(x-3)(x-4)$.

44. $(x-1)(x+1)(x-2)(x+2)(x-3)(x+3)$.

45. $(a+b+c+d)(a+b+c+d)(a+b+c+d)$.

46. $(a-b+c-d)(a-b+c-d)(a-b+c-d)$.

47. $(x^3-3xy^2+4y^2z-a)(x-a)(x-z)(x-2y)$.

48. $(2x^4-3x^3+4x^2-5x+6)$ by $(7x^2+8x-9)$.

49. $(x^3-3x^2y+3xy^2-y^3)$ by $(x^3-3x^2y+3xy^2-y^3)$.

50. $(x^4+4x^3y+6x^2y^2+4xy^3+y^4)$ by $(x^2+2xy+y^2)$.

51. $(\frac{1}{2}x^3-\frac{1}{3}x^2+\frac{2}{3}x+\frac{1}{4})$ by $(\frac{3}{2}x^4-\frac{2}{3}x^3-\frac{3}{4}x^2+\frac{1}{6}x-\frac{5}{6})$.

52. $(abc-3abd+6abx-3bdx)$ by $(2bac+4bxa-2xbd)$.

53. $(2^4a^4-2^3a^3\,3b+2^2a^2\,3^2b^2-2a\,3^3b^3+3^4b^4)$ by $(2a+3b)$.

54. $(5x^4-7x^3+8x^2-3x-5)$ by $(3x^5-4x^3-7x^4-9x^2-10x^3+3x-5)$.

55. $(a^2+2ab+2ac+2ad+2bc+2bd+2cd+b^2+c^2+d^2)$ by $(a+b+c+d)$.

56. $(.75x^4+.125x^3-.25x^2-.375)$ by $(.25x^4-1.2x^3+2.0\frac{1}{4}x^2-5.5x-.3)$.

57. $(2\frac{1}{3}x^4-3\frac{2}{3}x^3+4\frac{1}{5}x^2-\frac{3}{4}x+\frac{2}{5})$ by $(\frac{3}{7}x^3-\frac{3}{11}x^2-\frac{5}{21}x+\frac{5}{2})$.

Section IV

DIVISION

101. Division is the process of finding one of two factors when their product and the other factor are given.

102. The **Dividend** is the given product.

103. The **Divisor** is the given factor.

104. The **Quotient** is the required factor.

105. Proof: the divisor $\times$ the quotient $=$ the dividend.

LAWS FOR DIVISION

106. *A.* **Law of Order.** Successive divisions may be performed in any order.

ILLUSTRATION. $a \div b \div c \equiv a \div c \div b$.

For if $a = 16$, $b = 4$, $c = 2$,

then $$16 \div 4 \div 2 \equiv 16 \div 2 \div 4;$$
$$2 \equiv 2.$$

107. Law A for multiplication and law A for division may be **combined into one law** and stated as follows:

A succession of multiplications and divisions may be performed in any order.

ILLUSTRATION. $a \times b \div c \equiv a \div c \times b$.

For if $a = 8$, $b = 2$, $c = 4$,

then $$8 \times 2 \div 4 \equiv 8 \div 4 \times 2;$$
$$4 \equiv 4.$$

108. *B.* **Law of Grouping.** Dividing by two or more quantities in succession gives the same result as dividing by their product.

ILLUSTRATION. $a \div b \div c \equiv a \div (bc)$.

For, if $a = 16,\ b = 4,\ c = 2,$

then
$$16 \div 4 \div 2 \equiv 16 \div (4 \times 2);$$
$$2 \equiv 2.$$

109. *C.* **Law of Distribution.** The quotient of one algebraic expression divided by another is equal to the sum of the quotients obtained by dividing the terms of the first expression by the second.

ILLUSTRATION. $\dfrac{a+b+c}{x} \equiv \dfrac{a}{x} + \dfrac{b}{x} + \dfrac{c}{x}.$

For, if $a = 400,\ b = 60,\ c = 8,\ x = 2,$

then
$$\tfrac{468}{2} \equiv \tfrac{400}{2} + \tfrac{60}{2} + \tfrac{8}{2};$$
$$234 \equiv 200 + 30 + 4;$$
$$234 \equiv 234.$$

110. *D.* **Law of Signs.** When the sign of the divisor is *like* the sign of the dividend, the sign of the quotient is *plus;* when *unlike*, *minus*.

ILLUSTRATION. $+a^2 \div +a$, and $-a^2 \div -a = +a$; $+a^2 \div -a$, and $-a^2 \div +a = -a$.

Let the student state the reasons from the definition of multiplication.

111. *E.* **Law of Exponents.** The exponent of the quotient of two powers of the same quantity is equal to the exponent in the dividend minus the exponent in the divisor.

ILLUSTRATIONS. 1. $a^m \div a^n = a^{m-n}$. 2. $3^8 \div 3^2 = 3^6$.

Dem. Converse of that of multiplication. Why?

112. Cor. 1. Any quantity affected with the exponent zero is equal to one; for example, $a^0 = 1$.

Dem. 1. $a^m \div a^m = 1$,

$\because$ a quantity divided by itself equals unity.

2. $a^m \div a^m = a^0$, by law E.

3. $\therefore\ a^0 = 1$,

$\because$ two things equaling the same thing are equal to each other.

Let the student interpret the following diagram:

$$\begin{array}{llcc} 1. & \frac{7^3}{7^3} = 1 & \searrow & \\ & \qquad\ \ \parallel & & \frac{7^3}{7^3}. \\ 2. & \frac{7^3}{7^3} = 7^0 & \nearrow & \end{array}$$

113. Cor. 2. When both dividend and divisor are multiplied or divided by the same number, the quotient remains the same. Give an illustration.

114. Cor. 3. When the dividend is multiplied or divided by any integral number, the quotient is multiplied or divided by that number. Give an illustration.

115. Cor. 4. When the divisor is multiplied or divided by any integral number, the quotient is divided or multiplied by that number. Give an illustration.

116. **Cancellation** is the process of dividing both dividend and divisor (or numerator and denominator) by the same number, by striking out one or more factors common to both of them. See Cor. 2.

Note. Factors *only* are canceled; terms, as such, *never*. Further comment will be made under the subject of Fractions.

MENTAL EXERCISES

Perform the operations indicated:

1. -4×2.
2. $-6 \div -3$.
3. $+8 \times -4$.
4. $-27 \div +9$.
5. $+81 \div -2$.
6. $-5 \div -40$.
7. -6×-7.
8. $+0 \div +5$.
9. $-0 \times +5$.
10. $+0 \times +0$.
11. $-5 \times +0$.
12. $-4 \div \frac{2}{3}$.
13. $\frac{2}{3} \div -4$.
14. $-\frac{5}{6} \div -\frac{5}{2}$.
15. $-\frac{2}{5} \times -.6$.
16. $-\frac{0}{8} \times 3$.
17. $-8 \div 4 \times -2$.
18. $-9 \times -3 \div -6$.
19. $16 \div -2 \times -3 \div 5$.
20. $-32 \div 2 \div -2 \div 4$.
21. $42 \div -3 \div -7 \times -2$.
22. $-7 \times 0 \div -8 \times -11$.
23. $2 \times -\frac{1}{2} \div \frac{1}{3} \div -\frac{1}{4} \times 6 \div -2\frac{1}{2}$.
24. $-16 \times \frac{1}{4} \div -4 \times -\frac{5}{8} \div -\frac{1}{4}$.
25. $(-5+4) \div -1 \times (\frac{2}{3} - 1 + 0) \div -\frac{1}{3}$.
26. $-\frac{1}{2} \div \frac{1}{3} \times \frac{1}{4} \div \frac{1}{5} \times -8 \div -15$.
27. $-\frac{0}{5} \times -6 + 2 \times -3 - \frac{1}{3} \times -\frac{5}{7} \div \frac{1}{7}$.

117. PROBLEM 1. **To divide one monomial by another.**

Rule. *Determine the sign of the quotient by the Law of Signs. Divide the numerical coefficients, and to this quotient affix the quotient of the literal factors obtained by the Law of Exponents.*

Dem. See laws D and E.

MODEL SOLUTION

Divide $15\,a^4b^3m^4x^3y^2$ by $-5\,a^3bx^2y^2$.

$$\begin{array}{r|l} -5\,a^3bx^2y^2 & 15\,a^4b^3m^4x^3y^2 \\ \hline & -3\,ab^2m^4x \end{array}$$

Explanation. Since the signs are unlike, the sign of the quotient is $-$ by the Law of Signs. Dividing 15 by 5 gives 3 for the numerical coefficient of the quotient. Dividing the literal factors in accordance with the Law of Exponents, $a^4 \div a^3 = a$; $b^3 \div b = b^2$; m^4 may be treated as though divided by $m^0 (=1)$, giving a quotient m^4; $x^3 \div x^2 = x$; $y^2 \div y^2 = y^0 = 1$. Writing these results in succession gives the required quotient $-3\,ab^2m^4x$. A sufficient proof for this result is that $-5\,a^3bx^2y^2 \times -3\,ab^2m^4x = 15\,a^4b^3m^4x^3y^2$.

MENTAL EXERCISES

Divide:

1. $10\,a^3b^2k^5$ by $-5\,a^2bk^3$.
2. $-17\,x^2yz^3$ by xyz.
3. $-6\,abc$ by $-abc$.
4. $5\,d^5f^6g^7$ by $-2\,df^2g^3$.
5. $abxyz$ by $-xyz$.
6. $3\,a^{2n}bh^3$ by $4\,a^nh^n$.
7. $\frac{1}{2}\,kl^sm^tn$ by $\frac{2}{3}\,klmn$.
8. $\frac{5}{7}\,p^2q^3r^5$ by $\frac{3}{4}\,p^nq^sr^5$.
9. $(a+b)^2$ by $(a+b)$.
10. $21(x-y)^5 + 3$ by $(x-y)^3$.
11. $10(m-n)^3$ by $5(m-n)^2$.
12. $16\,a^2b(a^2-b^2)^4$ by $4\,a(a^2-b^2)^3$.
13. $\frac{3}{5}\,x^2y(p+q)^{3n}$ by $\frac{2}{3}(p+q)^n$.
14. $91\,km^n(2-3)^5$ by $7\,m(2-3)$.
15. $(a+b)(a-b)$ by $(a-b)$.
16. $51(a+b)(a-b)$ by $17(a+b)$.
17. $156(a+b)^3(a-b)^7$ by $13(a+b)(a-b)^5$.
18. $(a+b)(x-y)(m+n)$ by $x-y$.
19. $15(a+b+c)^4$ by $-3(a+b+c)^3$.
20. $(c-d)(f+g) \div (c-d)$ by $f+g$.
21. Prove Ex. No. 17 when $a=3$, $b=2$.
22. Prove Ex. No. 13 when $n=1$, $p=2$, $q=3$.

118. PROBLEM 2. **To divide a polynomial by a monomial.**

Rule. *Divide each term of the polynomial by the monomial, writing the separate results in succession with their own signs.*

Dem. See law C.

Model Solution

Divide $91\,a^5m^3y - 56\,a^3m^sx + 49\,a^4m^7$ by $-7\,a^3m^2$.

$$\begin{array}{r|l} -7\,a^3m^2 & 91\,a^5m^3y - 56\,a^3m^sx + 49\,a^4m^7 \\ \hline & -13\,a^2my + 8\,m^{s-2}x - 7\,am^5 \end{array}$$

Since by law C the quotient of the sum equals the sum of the separate quotients, each term of the polynomial may be divided by $-7\,a^3m^2$, giving for the quotient $-13\,a^2my + 8\,m^{s-2}x - 7\,am^5$.

MENTAL EXERCISES

Divide:

1. $6\,x^2y - 10\,x^3y^2 - 12\,x^2yz$ by $2\,x^2y$.
2. $a - a^2 + a^3 - a^n$ by a.
3. $27\,a^3x^2 - 18\,ax^2$ by $-9\,ax^2$.
4. $a(x-y) - b(x-y)$ by $x - y$.
5. $25\,a^2x - 10\,a^3x^2 - 15\,amx + 5\,ax$ by $5\,ax$.
6. $126\,ax^2y^3 - 18\,a^2x^3y^5 + 24\,a^3x^2y^4$ by $6\,ax^2y$.
7. $5\,a^{2m} + 10\,a^mb^m - 15\,a^{3m}$ by $5\,a^m$.
8. $a^{2m}b^{3m+1} - a^{3m+1}b^{2m}$ by $a^{2m}b^{2m}$.
9. $\frac{1}{2}\,a^2 - \frac{1}{3}\,a^4b + \frac{1}{4}\,a^5x$ by $\frac{1}{12}\,a^2$.
10. $\frac{5}{7}\,x^2y^3 - \frac{2}{3}\,x^3y^5z + \frac{2}{21}\,x^4y^4z^2$ by $\frac{2}{21}\,x^2y^3$.
11. $\frac{2}{3}\,abc - \frac{3}{4}\,a^2b^2c^2 + \frac{5}{6}\,a^nb^m$ by $\frac{30}{12}\,a^rb^s$.
12. $y^{n-r} - y^{s+5} + y^{s-t+u}$ by $\frac{1}{2}\,y^3$.
13. $5^3 - 25 + 2\cdot 5^n - a\cdot 5^2$ by $5^2\cdot a^0$.

119. PROBLEM 3. **To divide a polynomial by a polynomial.**

Rule 1. *Arrange the dividend and divisor with reference to the same letter.*

2. *Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.*

3. ***Multiply all the divisor by this term of the quotient, and subtract the product from the dividend.***

4. ***To this remainder annex as many terms of the dividend as may be necessary to form a new dividend. Divide as before, and continue the process until all the terms of the dividend have been used.***

Dem. See law C.

MODEL SOLUTION NO. 1

Divide $x^4-50xy^3+35x^2y^2+24y^4-10x^3y$ by x^2+8y^2-6xy.

$$\begin{array}{rrrrr|ll}
\multicolumn{5}{c|}{\text{Dividend.}} & & \\
x^4 & -10x^3y & +35x^2y^2 & -50xy^3 & +24y^4 & x^2-6xy+8y^2, & \text{Divisor.} \\
x^4 & -6x^3y & +8x^2y^2 & & & x^2-4xy+3y^2, & \text{Quotient.} \\
\hline
 & -4x^3y & +27x^2y^2 & -50xy^3 & & & \\
 & -4x^3y & +24x^2y^2 & -32xy^3 & & & \\
\hline
 & & 3x^2y^2 & -18xy^3 & +24y^4 & & \\
 & & 3x^2y^2 & -18xy^3 & +24y^4 & & \\
\hline
\end{array}$$

Explanation. The dividend and divisor are first both arranged by law A according to the descending powers of x, and are kept that way throughout. Since the term of highest degree of x in the dividend, *i.e.*, x^4, must be the product of the terms containing the highest degree of x in the divisor and quotient, the *first term* in the quotient must be $x^4 \div x^2 = x^2$. Now, as it has been found that $x^2-6xy+8y^2$ is contained in the dividend x^2 times (and more), x^2 times the divisor may be taken out of the dividend, and then may be found how many times the divisor is contained in what is left of the dividend. Multiplying all the divisor by x^2, and subtracting the result from the dividend, leave the remainder $-4x^3y+27x^2y^2-50xy^3+24y^4$. Since this remainder must be the product of the divisor by the other terms of the quotient, the first term of the remainder must be the product of the first term of the divisor by the next term of the quotient. Therefore the *second term* of the quotient is $-4x^3y \div x^2 = -4xy$. Multiplying all the divisor by $-4xy$, and subtracting as before, leave the remainder $3x^2y^2 - 18xy^3 + 24y^4$. By the same course of reasoning the *third term* of the quotient is found to be $3y^2$. Since there has been subtracted in succession the divisor multiplied by x^2, by $-4xy$, and by

$3\,y^2$, there has been subtracted altogether the product of the divisor and the quotient. There being no remainder, the quotient is $x^2 - 4\,xy + 3\,y^2$.

Note. The connection between law C and an example in long division may be shown by the separate divisions of the example just explained. Thus,

$$\frac{x^4 - 10\,x^3y + 35\,x^2y^2 - 50\,xy^3 + 24\,y^4}{x^2 - 6\,xy + 8\,y^2} = \frac{x^4 - 6\,x^3y + 8\,x^2y^2}{x^2 - 6\,xy + 8\,y^2}$$

$$+\frac{-4\,x^3y + 24\,x^2y^2 - 32\,xy^3}{x^2 - 6\,xy + 8\,y^2} + \frac{3\,x^2y^2 - 18\,xy^3 + 24\,y^4}{x^2 - 6\,xy + 8\,y^2} = x^2 - 4\,xy + 3\,y^2.$$

Model Solution No. 2

Divide $27\,a^3 - 8\,b^3$ by $3\,a - 2\,b$.

$$\begin{array}{l|l}
27\,a^3 + 0\,a^2b + 0\,ab^2 - 8\,b^3 & 3\,a - 2\,b \\
\underline{27\,a^3 - 18\,a^2b} & \overline{9\,a^2 + 6\,ab + 4\,b^2} \\
\quad 18\,a^2b + 0\,ab^2 & \\
\quad \underline{18\,a^2b - 12\,ab^2} & \\
\qquad 12\,ab^2 - 8\,b^3 & \\
\qquad \underline{12\,ab^2 - 8\,b^3} &
\end{array}$$

Model Solution No. 3

Divide $a^3 + b^3 + c^3 - 3\,abc$ by $a + b + c$.

$$\begin{array}{l|l}
a^3 + b^3 + c^3 - 3\,abc & a + b + c \\
\underline{a^3 + a^2b + a^2c} & \overline{a^2 - ab + b^2 - ac - bc + c^2} \\
\quad -a^2b - a^2c + b^3 + c^3 - 3\,abc & \\
\quad \underline{-a^2b - ab^2 \qquad\qquad - abc} & \\
\qquad ab^2 + b^3 - a^2c - 2\,abc + c^3 & \\
\qquad \underline{ab^2 + b^3 + b^2c} & \\
\qquad\quad -a^2c - 2\,abc - b^2c + c^3 & \\
\qquad\quad \underline{-a^2c - abc - ac^2} & \\
\qquad\qquad -abc + ac^2 - b^2c + c^3 & \\
\qquad\qquad \underline{-abc \qquad - b^2c - bc^2} & \\
\qquad\qquad\quad ac^2 + bc^2 + c^3 & \\
\qquad\qquad\quad \underline{ac^2 + bc^2 + c^3} &
\end{array}$$

MODEL SOLUTION NO. 4

Divide $x^{4n}+x^{2n}y^{2n}+y^{4n}$ by $x^{2n}+x^ny^n+y^{2n}$.

$$
\begin{array}{rrrrr|l}
x^{4n} & +\,x^{2n}y^{2n} & +\,y^{4n} & & & x^{2n}+x^ny^n+y^{2n} \\
\cline{6-6}
x^{4n}+x^{3n}y^n & +\,x^{2n}y^{2n} & & & & x^{2n}-x^ny^n+y^{2n} \\
\cline{1-2}
-\,x^{3n}y^n & & & & & \\
-\,x^{3n}y^n & -\,x^{2n}y^{2n} & -\,x^ny^{3n} & & & \\
\cline{1-3}
 & x^{2n}y^{2n} & +\,x^ny^{3n} & +\,y^{4n} & & \\
 & x^{2n}y^{2n} & +\,x^ny^{3n} & +\,y^{4n} & & \\
\cline{2-4}
\end{array}
$$

MODEL SOLUTION NO. 5

Divide $\frac{2}{15}a^2-\frac{17}{50}ab-\frac{3}{5}b^2$ by $\frac{2}{3}a+\frac{4}{5}b$.

$$
\begin{array}{lll|l}
\frac{2}{15}a^2 & -\,\frac{17}{50}ab & -\,\frac{3}{5}b^2 & \frac{2}{3}a+\frac{4}{5}b \\
\cline{4-4}
\frac{2}{15}a^2 & +\,\frac{4}{25}ab & & \frac{1}{5}a-\frac{3}{4}b \\
\cline{1-3}
 & -\,\frac{1}{2}ab & -\,\frac{3}{5}b^2 & \\
 & -\,\frac{1}{2}ab & -\,\frac{3}{5}b^2 & \\
\cline{2-3}
\end{array}
$$

MODEL SOLUTION NO. 6

Divide $x^3-(a-b+c)x^2-(ab-ac-bc)x+abc$ by $x-c$.

$$
\begin{array}{llll|l}
x^3 & -(a-b+c)x^2 & -(ab-ac+bc)x & +\,abc & x-c \\
\cline{5-5}
x^3 & -\qquad cx^2 & & & x^2-(a-b)x-ab \\
\cline{1-4}
 & -(a-b)x^2 & -(ab-ac+bc)x & & \\
 & -(a-b)x^2 & -(\quad\ -ac+bc)x & & \\
\cline{2-3}
 & & -\,abx & +\,abc & \\
 & & -\,abx & +\,abc & \\
\cline{3-4}
\end{array}
$$

EXAMPLES

Divide, and verify each quotient:

1. $a^2+2ab+b^2$ by $a+b$.
2. $a^2-2ab+b^2$ by $a-b$.
3. a^2-b^2 by $a-b$.
4. a^2-b^2 by $a+b$.

5. $a^3 - b^3$ by $a - b$.

6. $a^3 + b^3$ by $a + b$.

7. $a^4 - b^4$ by $a - b$.

8. $a^4 - b^4$ by $a^2 + b^2$.

9. $a^5 - b^5$ by $a - b$.

10. $a^6 + b^6$ by $a^2 + b^2$.

11. $x^{21} + y^{21}$ by $x^7 + y^7$.

12. $a^6b^{12} - 64$ by $ab^2 - 2$.

13. $16k^4 - 81l^4$ by $2k - 3l$.

14. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.

15. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

16. $x^2 + (a + b)x + ab$ by $x + a$.

17. $8c^2 - 6cd - 5d^2$ by $4c - 5d$.

18. $x^3 + y^3 + 3xy - 1$ by $x + y - 1$.

19. $(ab + cd)^2 - 1$ by $(ab + cd) + 1$.

20. $acx^2 + (ad + bc)x + bd$ by $ax + b$.

21. $x^6 + 2x^4 + 5x^2 - 1$ by $x^3 - 2x^2 + 3x - 1$.

22. $25y^4 - 36y^2z^2 + 4z^4$ by $5y^2 - 4yz - 2z^2$.

23. $x^4 - 10x^3 + 39x^2 - 70x + 49$ by $x^2 - 5x + 7$.

24. $2x^2 - 16x^6 - 7x^4 + 1$ by $1 + 2x + 4x^3 + 3x^2$.

25. $4x^4 - 12x^3 - 7x^2 + 24x + 16$ by $2x^2 - 3x - 4$.

26. $17ac - 3b^2 - 11ab + 4a^2 + bc + 4c^2$ by $a - 3b + 4c$.

27. $13x^2a^2 - 13x^3a - 13xa^3 + 6x^4 - 5a^4$ by $2x^2 - 3xa - a^2$.

28. $2x^4 - 13x^3a + 31x^2a^2 - 38xa^3 + 24a^4$ by $x^2 - 5xa + 6a^2$.

29. $x^6 - 5x^5 + 15x^4 - 24x^3 + 27x^2 - 23x + 9$ by $x^2 - 2x + 1$.

30. $4y^6 - 24y^5 + 60y^4 - 80y^3 + 60y^2 - 24y + 4$ by $2y^2 - 4y + 2$.

31. $10a^4 + 34a^2x^2 - 27a^3x - 18ax^3 - 8x^4$ by $2a^2 - 3ax + 4x^2$.

32. $x^{2n} + 2x^ny^n + y^{2n}$ by $x^n + y^n$.

33. $x^3 - \frac{13}{12}x^2 + \frac{3}{8}x - \frac{1}{24}$ by $x - \frac{1}{4}$.

34. $x^3 + \frac{17}{4}x^2 - \frac{79}{8}x + \frac{15}{4}$ by $x - \frac{5}{4}$.

35. $x^6 - 2x^3y^3 + y^6$ by $x^2 - 2xy + y^2$.

36. $a^3 - b^3 + 9ab + 27$ by $a - b + 3$.

37. $4(x - y)^3 - x + y$ by $2x - 2y - 1$.

38. $\frac{1}{3}x^3 - \frac{17}{36}x^2 + \frac{1}{3}x - \frac{1}{8}$ by $\frac{2}{3}x - \frac{1}{2}$.

39. $\frac{1}{3}x^3 + \frac{17}{6}x^2 - \frac{5}{4}x + \frac{9}{4}$ by $\frac{1}{3}x + 3$.

40. $x^{4n} + x^{2n}y^{2n} + y^{4n}$ by $x^{2n} - x^ny^n + y^{2n}$.

41. $x^{m+1} + x^my + xy^m + y^{m+1}$ by $x^m + y^m$.

42. $49a^2b^2 - \frac{28}{3}ab^3 + \frac{4}{9}b^4$ by $7ab - \frac{2}{3}b^2$.

43. $c^{3m} - 3c^{2m}d^n + 3c^md^{2n} - d^{3n}$ by $c^m - d^n$.

44. $32s^5 - 16ts^4 - 2t^4s + t^5$ by $4s^2 - 4ts + t^2$.

45. $x^6 + 1008x + 720$ by $x^3 - 6x^2 + 12x + 12$.

46. $5x^{3n} - 15x^{2n}y^n + 15x^ny^{2n} - 5y^{3n}$ by $x^n - y^n$.

47. $x^3 + (y + z)x^2 - xyz - yz(y + z)$ by $x^2 - yz$.

48. $1 - 2x - 31x^2 + 72x^3 - 30x^4$ by $1 - 6x + 3x^2$.

49. $2x^5 - 18x^4 + 39x^3 - 25x^2 + x + 1$ by $2x^2 - 4x + 1$.

50. $x^5 - 4x^2 + x^3 - 2x + 4 + 0x^4$ by $x^3 + 0x^2 + 2x - 4$.

51. $15x^2 + 9x + 13xy - 6y^2 + 20y - 6$ by $5x + 6y - 2$.

52. $3v^2 + 8vw + 4w^2 + 10vz + 8wz + 3z^2$ by $3z + 2w + v$.

53. $4a^5 - 16a^3b^2 + 10a^2b^3 + 15ab^4 - 25b^5$ by $2a^3 - 3ab^2 + 5b^3$.

54. $a^6 + 13a^4x^2 - 6a^5x - 12a^3x^3 + 4a^2x^4 - 36x^6$ by $a^2 - 2ax - 3x^2$.

55. $hkx^4 + 2(h - k)x^3 - (h^2 + 4 - k^2)x^2 + 2(h + k)x - hk$ by $kx^2 + 2x - h$.

56. $12a^4 - 22a^3b + 31a^2b^2 - 14ab^3 + 5b^4$ by $3a^2 - 4ab + 5b^2$.

57. $15a^4 + 16a^3 - 30a^2 - 13a + 20$ by $5a^3 - 3a^2 - 5a + 4$.

58. $x^{12} - x^9y^3 + x^6y^6 - x^3y^9 + y^{12}$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

59. $4a^5 - 8a^4x + 27a^3x^2 - 28a^2x^3 + 36ax^4 - 7x^5$ by $2a^2 + 7x^2 - ax$.

Symbols of Aggregation Involving Multiplication and Division

Simplify by removing symbols of aggregation and combining:

1. $x(2a+b)(2a-b)-3\{(a-b)2c\}x$. Check.

2. $6a-[3a-\{2c(m-n)-5c(m+n)+3a\}+4cm]$. Check.

3. $[a(x+y)(x-y)-\{2axy-(axy-2)\}+3]-(ax^2-ay^2)$.

4. $bc(c^2-b^2)-\{(b-c)(b+c)bc\}-[\{(3a-x)-5\}+3]$.

5. $(x+y)^2-(x-y)^3+(x+y)(x-y)+(x+y)^3-\{a-(2a+4)\}$.

6. $\{(1+x^3)-(1-x^3)\}\{(-5-3a)-2a\}-\overline{2cx(x^2-y^2)}$.

7. $bc(b^2-c^2)+ac(c^2-a^2)+ab(a^2-b^2)-(a+b+c)\{a^2(b-c)$ $+b^2(c-a)+c^2(a-b)\}$.

8. $3(x+1)^3-x^3-3(x+2)^3+(x+3)^3$. Check.

9. $5(x+y+z)\{x-y+z\}[-x+y-z]+\overline{6x^3+5y^3+5z^3}$.

10. $\{\overline{6-5a}(2+2a)-3\overline{ca-b}+\overline{5bc-7ac}\}\div 2$.

11. If $x=2a-3b^2$, find the value of $x^3+9b^6-8a^3$.

12. If $x=3$, find the value of $x^3-2\{x^2-(x-3)\}$.

13. If $x=\frac{1}{2}a$, find the value of $x^3+2x^2-3(x^2-\overline{x-1})$.

14. If $x=a^2-ab+b^2$ and $y=a^2+ab+b^2$, find the value of x^2-xy-y^2.

Simplify, then arrange and inclose in parentheses the coefficients of the unknown quantities:

Model Solution

$$
\begin{aligned}
&2a(3x-2mx-y)-x(3a+3am+7)-(3y+5)-2xy+axy\\
&=6ax-4amx-2ay-3ax-3amx-7x-3y-5-2xy+axy\\
&=6ax-4amx-3ax-3amx-7x-2ay-3y-2xy+axy-5\\
&=3ax-7amx-7x-2ay-3y-2xy+axy-5\\
&=(3a-7am-7)x-(2a+3)y-(2-a)xy-5.
\end{aligned}
$$

15. $7c(4x-3cy-2)-y(2b-5c^2)-4ax(c-2)-15+11.$

16. $b(m+xy)c-5(m-xy+y^2)bc-2b(y+c)+(bc-x)(1+x)y.$

17. $3a(x-y)+2(x-y)^2-b(x+y)^2+\overline{2-3ax}.$

18. $(ax+bx-cx)^2-(5+2abx+c^2x^2)-(x^2-xa+a^2)(a+x).$

Simplify:

19. $x-(2b-x)-\dfrac{2ax-2bx}{a-b}.$ Check.

20. $3[a+b(3-a)]-\dfrac{3abx-9ax^2}{3ax}.$ Check.

21. $\dfrac{4x^2-10ax+8xy}{2x}-\dfrac{4x^2-25a^2}{2x-5a}-\dfrac{16y^2+24xy}{4y}.$

22. $5-\left\{\left(\dfrac{x^3+a^3}{x+a}-\dfrac{x^3-a^3}{x-a}\right)-\left(\dfrac{x^2-a^2}{x+a}-\dfrac{x^2-a^2}{x+a}\right)+5\right\}.$

23. $3a\left(\dfrac{4a^2-8ax+4x^2}{a-x}\right)-\dfrac{8a^2+2a}{2a}+2\{b+a(2-a)\}.$

24. $(x+4)\left(\dfrac{x^2-3xa-12a+4x}{x-3a}\right)-(x+2)(x+2)-\left(4a-\dfrac{8ab}{2b}\right).$

25. $x\left(\dfrac{x^2+x-12}{x+4}\right)-\dfrac{(7a-\{30ab-3a\})}{5a}-(x^2+3x).$

26. $\dfrac{a^4-b^4}{a^2+b^2}-(a^2+b^2)+\dfrac{a^6+b^6}{a^2+b^2}-(a^2-ab+b^2)(a^2+ab+b^2).$

27. $5\left(\dfrac{x^3-27}{x-3}\right)-\{2-(9-3x)(1-x)\}-\left\{\dfrac{(4a-x)(x+4a)}{7a-\overline{x+3a}}\right\}(4a-x).$

28. $\dfrac{(x^2-y^2)\{2x(x-y)\}}{4(x-y)^3(x^2+xy+y^2)x}\cdot\dfrac{2xy(x^3-y^3)}{x-y}.$

29. $\dfrac{27x^3+8}{9x^2-6x+4}\div\dfrac{9x^2-4}{3x-2}+(3x+2)^0(3x-2)^0.$

30. $(a^{4n}+a^{2n}b^{2n}+b^{4n})\div(a^{2n}-a^nb^n+b^{2n})$ when $a=1$, $b=2$, $n=\frac{1}{2}.$

SYNOPSIS FOR REVIEW, CHAPTER II

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CHAPTER III

LINEAR EQUATIONS, ONE UNKNOWN QUANTITY

DEFINITIONS

120. An **Equation** is a statement in mathematical symbols that one expression is equal to another.

ILLUSTRATIONS. $5 = 5$, $5 \times 2 = 10$, $a + 3a = 4a$, $6x = 12$, $2a - \frac{3}{5} + \frac{x+4}{7} = 8 + b$, are equations.

121. The **First Member** of an equation is the expression written before the sign of equality.

122. The **Second Member** of an equation is the expression written after the sign of equality.

123. **Identical Expressions** are equal expressions which are always true whatever be the values of the symbols of quantity used.

ILLUSTRATIONS. $8 \equiv 8$, $(x+a)(x-a) \equiv x^2 - a^2$, $a \equiv a$.

124. Equations may be divided into two classes: those of **Identity** and those of **Condition**.

An **Identical Equation** is a statement of equality between two identical expressions.

A **Conditional Equation** is one true only for particular values of the symbols of quantity used.

ILLUSTRATION. $3x = 12$ is a conditional equation, for it is true only for the particular value of x, which is 4.

NOTE. Identical equations will hereafter be called *identities*, and conditional equations simply *equations*.

125. A **Numerical Equation** is one in which all the known quantities are represented by decimal numbers.

ILLUSTRATION. $3x - 5 = 15 + 2$.

126. A **Literal Equation** is one in which some or all of the known quantities are represented by letters.

ILLUSTRATIONS. $ax + by + c = d$, all letters. $\frac{2}{3} + ax + y = b$, letters in part.

127. **Equivalent Equations** express *like* conditions of the same problem and *can* be reduced to the same form.

ILLUSTRATION. $2x + 5x = 7 + 14$, and $10x - 3x = 53 - 32$ are equivalent because both can be reduced to $7x = 21$.

128. **Independent Equations** express *different* conditions of the same problem and *cannot* be reduced to the same form.

ILLUSTRATION. $x + y = 5$, $2x + 3y = 13$.

129. An **Integral Equation** is one in which no unknown quantity appears in the denominator.

ILLUSTRATIONS. $5x + 3 = \frac{1}{2}x - 10$, $\frac{1}{3}(ax + b) = \frac{cx + 4}{k}$, and $\frac{x}{3} - \frac{bx}{c} = \frac{x - a}{h + g}$.

130. A **Fractional Equation** is one in which one or more unknown quantities appear in the denominator.

ILLUSTRATION. $\frac{x + 3}{x + 2} - 3 = a + \frac{b}{x} - \frac{3}{x + 1}$.

131. A **Rational Equation** is one that is free from radical signs and fractional exponents.

132. An **Irrational Equation** is one that contains fractional exponents or radical signs.

ILLUSTRATION. $x^{\frac{2}{3}} + ay = \sqrt{2}$.

133. The **Degree** of an equation is determined by the highest number of unknown factors occurring in any one term, after the equation has been freed from fractions, fractional exponents, and radical signs with respect to the unknown quantities.

ILLUSTRATIONS. $ax + by + c = 0$ is of the first degree. $Ax^2 + Bx + C = 0$ is of the second degree. $abxyz - 3xy + s = 0$ is of the third degree.

134. A **Linear Equation** is an equation of the first degree.

135. A **Quadratic Equation** is an equation of the second degree.

136. A **Cubic Equation** is an equation of the third degree.

137. A **Biquadratic Equation** is an equation of the fourth degree.

138. **Higher Equations** are equations above the second degree.

139. The **Main Problem** which arises in connection with each conditional equation is to find a value of the unknown quantity which will render the equation an identity, literal or numerical.

140. An equation is said to be **satisfied** when the substitution of the value of the unknown quantity renders the equation an identity.

141. A **Solution** of an equation of one unknown quantity is any value of the unknown quantity which satisfies the equation.

NOTE. The term is also applied to the method of obtaining any such value.

142. **Solutions** are of two kinds: (1) **Formal**, (2) **Algebraic.**

A **Formal Solution** is one in which the value of the unknown quantity is expressed by an arithmetical number.

1. $3x = 6;$	2. $2x = 1;$	3. $8x = 70;$
$x = 2;$	$x = \frac{1}{2} = .5;$	$x = 8\frac{3}{4} = 8.75;$
$\because\ 6 \equiv 6.$	$\because\ 1 \equiv 1.$	$\because\ 70 \equiv 70.$

An **Algebraic Solution** is one in which the value of the unknown quantity is expressed in an algebraic form containing only known quantities.

1. $ax = ab;$	2. $ax = bc;$	3. $x^2 = 2;$
$x = b;$	$x = \dfrac{bc}{a};$	$x = \pm\sqrt{2};$
$\because\ ab \equiv ab.$	$\because\ bc \equiv bc.$	$\because\ 2 \equiv 2.$

143. An **Approximate Numerical Solution** is one in which the value of the unknown quantity cannot be exactly obtained, but may be brought to any required degree of exactness.

ILLUSTRATION. $x = \sqrt{2}$ is approximately equivalent to $x = 1.4142$.

144. In connection with the solution of equations **two facts** should be borne in mind:

1. Every conditional equation is assumed to be an identity; *i.e.*, the unknown quantity is assumed to have such a value as will render the equation an identity.

2. The ultimate test of every solution is that the value of the unknown quantity shall satisfy the equation.

145. To **Verify** a solution is to substitute the supposed value of the unknown quantity to see whether it satisfies the equation.

146. A **Root** of an equation is the quantity which, substituted for the unknown quantity, satisfies the equation.

147. In general, the solution of an equation requires several transformations, which will therefore be the next subject to consider.

TRANSFORMATION OF EQUATIONS

148. To **Transform** an equation is to change the form of the equation without destroying the equality of its members.

149. (1) **Clearing of Fractions,** (2) **Transposition,** (3) **Collecting Terms, and** (4) **Dividing by the Coefficient** of the unknown quantity are the four principal transformations of linear equations.

The **Principal Axioms** used in the transformation of equations are as follows:

150. Axiom. (*a*) If equal quantities be multiplied or divided by the same or equal quantities, the products will be equal and the quotients will be equal.

Illustrations. If $a=b$, then $ad=bd$. If $a=b$, then $\frac{a}{c}=\frac{b}{c}$.

The former is used in clearing of fractions; the latter in dividing by the coefficient of the unknown quantity.

Note. Axiom (*a*) does not include the case of zero as a divisor. For if both members of $7\times 0=4\times 0$ are divided by zero, the equation becomes $7=4$.

151. Axiom. (*b*) If the same or equal quantities be added to or subtracted from equal quantities, the sums will be equal and the differences will be equal.

Illustrations. If $a=b$, then $a+t=b+t$. If $a=b$, then $a-t=b-t$. Used in transposing terms.

152. Axiom. (*c*) The whole is equal to the sum of all its parts.

Illustration. If $2x+3x+x=10+5+7+2$, then $6x=24$. Used in collecting terms.

For general use the above Axioms may be combined into the **two following Axioms**: the first dealing with both members, the second with one member or any term.

153. Axiom 1. Both members of an equation may be increased or diminished alike.

154. Axiom 2. Any operation may be performed upon either member or any term which does not affect the value of that member or term.

SOLUTION OF LINEAR EQUATIONS WITH ONE UNKNOWN QUANTITY

155. To **Solve by Inspection** is to assume a solution and test its accuracy by substitution.

Illustration. $4x-5=x+1$

becomes $4-5=1+1$ when it is assumed that $x=1$,

but $8-5=2+1$ when it is assumed that $x=2$.

Hence the root of the equation is 2.

156. PROBLEM 1. **To clear an equation of fractions.**

Rule. *Multiply both members by the lowest common multiple of all the denominators.*

Dem. This process does not destroy the equality. See Axiom 1. The equation is cleared of fractions because some factor in the L. C. M. cancels each denominator.

$$\frac{x-2}{3}-\frac{3x+4}{2}=\frac{2(x+3)}{3}-\frac{3(2x-1)}{4}+\frac{1}{6};$$

$$4x-8-18x-24=8x+24-18x+9+2.$$

EXPLANATION. The L. C. D. is 12. Multiply the first fraction by 12 by canceling the factor 3 from 12 and the denominator. Multiplying the numerator $x-2$ by the remaining factor of 12 gives $4x-8$. In the same manner 12 times the second fraction is $18x+24$. But the sign minus before the fraction signifies that this result is a subtrahend, *i.e.*, to be subtracted. Hence the signs of $18x+24$ must be changed, and the expression becomes $-18x-24$. Multiplying the third fraction by 12 gives $8(x+3)$ or $8x+24$. The fourth and fifth fractions become $-18x+9$ and 2 respectively.

157. PROBLEM 2. **To transpose a term.**

Rule. *Add to both members of the equation an expression numerically equal to the terms to be transposed, but with opposite sign.*

Dem. This process does not destroy the equality. See Axiom 1.

NOTE. It is convenient to say a term has been transposed when in reality it has not been transposed, but destroyed. It has the appearance of being transposed. Thus —

$$4x-8-18x-24=8x+24-18x+9+2$$

becomes $4x-18x-8x+18x=8+24+24+9+2$, by transposition.

Explanation. Adding $+8$ to both members destroys the -8 in the first member, and causes $+8$ to appear in the second member. Adding $+24$ to both members destroys the -24 in the first member, and causes $+24$ to appear in the second member. Adding $-8x$ and also $+18x$ to both members destroys the $8x$ and the $-18x$ in the second member, and causes $-8x$ and $+18x$ to appear in the first member. All the unknown terms are now in the first member, and all the known terms are in the second member.

158. Collecting Terms is the process of combining the terms of each member into the simplest form.

Illustration. $4x-18x-8x+18x=8+24+24+9+2$

becomes $-4x=67$ by collecting terms. By what Axiom?

159. Dividing by the Coefficient of the unknown quantity is the process of dividing both members of the equation by the coefficient.

Illustration. $-4x=67$ becomes $x=-\frac{67}{4}$ by dividing by coefficient of x. By what Axiom?

160. Cor. All the signs of an equation may be changed by multiplying or dividing both members by -1.

Illustration. $-3x+x-7=3-8+4$

becomes $3x-x+7=-3+8-4.$

How? Is equality changed? Why?

161. Prop. A linear equation has one, and only one, root.

Dem. By the first three transformations any linear equation may be reduced to the form (1) $Ax=B$. By the fourth transformation, (2) $x=\frac{B}{A}$. Since (1) is equivalent to (2), $\frac{B}{A}$ contains the roots of (1). But $\frac{B}{A}$ gives but one result, and hence one root; therefore $Ax=B$ has but one root. Hence the original equation has but one root.

162. PROBLEM 3. **To solve a linear equation by the four transformations.**

Rule 1. *Clear of fractions.*

Dem. See Axiom 1; also Problem 1.

2. *Transpose all terms containing the unknown quantity to the first member, and all others to the second member.*

Dem. See Axiom 1; also Problem 2.

3. *Collect terms.*

Dem. See Axiom 2.

4. *Divide both members by the coefficient of the unknown quantity.*

Dem. See Axiom 1.

MODEL SOLUTION

Solve $$\frac{3x}{2}-\frac{\frac{3}{2}(x+3)}{12}=\frac{2}{3}-\frac{7x}{24}+\frac{5x}{4}.$$

1. $36x-3x-9=16-7x+30x$, by clearing fractions. Axiom 1.
2. $36x-3x+7x-30x=9+16$, by transposing. Axiom 1.
3. $10x=25$, by collecting terms. Axiom 2.
4. $x=\frac{5}{2}$, by dividing by coefficient. Axiom 1.

Verification

a. $\dfrac{\frac{15}{2}}{2}-\dfrac{\frac{3}{2}(\frac{5}{2}+3)}{12}=\dfrac{2}{3}-\dfrac{\frac{35}{2}}{24}+\dfrac{\frac{25}{2}}{4}$, by substituting $\frac{5}{2}$ for x in (1).

b. $\frac{15}{4}-\frac{11}{16}=\frac{2}{3}-\frac{35}{48}+\frac{\;}{8}$, by simplifying (*a*).

c. $\frac{49}{16}\equiv\frac{49}{16}$, by collecting terms in (*b*).

Note. To verify any root, *always* substitute it in the original equation. Why?

EXAMPLES

Solve for the unknown quantity, and verify:

1. $\frac{2s-3}{15}-\frac{3s+4}{20}+\frac{5-s}{12}-\frac{3}{5}=-\frac{47}{60}$.

2. $\frac{6t+7}{12}+\frac{8-t}{3}-\frac{5t-9}{18}=2t-\frac{17}{36}$.

3. $\frac{3Q}{14}-\frac{5+Q}{56}-\frac{\frac{1}{2}Q-7}{14}=\frac{5}{28}-\frac{1}{2}Q$.

4. $\frac{1}{2}(v-5)+\frac{1}{3}(5-v)+\frac{1}{4}(v+7)=\frac{5}{6}(v+2)$.

5. $\frac{2}{3}\{w-(3w-\frac{3}{2})\}-\frac{5}{7}(3\overline{w-2}-w)=0$.

6. $\frac{7}{5}-\frac{32}{5}-\left(2z-\frac{3z-7}{14}\right)=-\frac{7z+5}{3}$.

7. $\frac{2x}{7}-\frac{5x}{91}-\frac{3x-1}{13}+\frac{2x}{7}=\frac{1}{2}(x-1)-11$.

8. $3y-\frac{5y+2}{27}-\frac{6-3y}{9}-\frac{3}{18}+\frac{y}{\frac{3}{2}}=0$.

9. $\frac{u-2}{26}-\frac{2-u}{2}=-\frac{3-4u}{13}$.

10. $2x-[4-2\{3x+(8-x)\}]=\frac{7x+3}{5}$.

11. $\frac{7}{8}x-\frac{3}{7}x+1\frac{1}{8}=2\frac{1}{4}x+\frac{9x}{14}-20\frac{25}{28}$.

12. $3x-\frac{x-4}{4}-4=\frac{5x+14}{3}$.

13. $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}$.

14. $\frac{2t+1}{3}-\frac{4t+5}{4}=\frac{2t+5}{8}-\frac{t+8}{6}$.

15. $\frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36$.

16. $\frac{7x+5}{3}-\left(2x-\frac{3x-7}{14}\right)-5=0.$

17. $\frac{3x+1}{5}-\frac{7x+2}{10}+\frac{3x}{4}-\frac{7x}{8}+12=0.$

18. $5z-[8z-3\{16-6z-(4-5z)\}]=6.$

19. $\frac{2x-5}{9}-\frac{x+3}{5}-\frac{x}{3}=2x-15\frac{7}{8}.$

20. $\frac{2-3x}{1.5}+\frac{5x}{1.25}-\frac{2x-3}{9}=\frac{x-2}{1.8}+2\frac{7}{9}.$

21. $3.3x-\frac{.72x-.55}{.5}=.1x+9.9.$

22. $\frac{3}{8}x+\frac{5}{6}(11-x)=1-\frac{5}{24}(x-2).$

23. $\frac{1}{8}(4x-3)=\frac{1}{12}(2x-3)-\frac{1}{6}(3x-2).$

24. $10\frac{1}{2}(x-\frac{2}{3})+\frac{1}{2}(3\frac{3}{4}x+7)=21\frac{1}{4}.$

25. $\frac{5x+2}{3}-\left(3-\frac{3x-1}{2}\right)=\frac{3x+19}{2}-\left(\frac{x+1}{6}+3\right).$

MENTAL EXERCISES

1. How many hours will it take to walk s miles at the rate of $\frac{7}{2}$ miles an hour? Of Q miles an hour? Of 9 miles in $1\frac{1}{2}$ hours?

2. What must be added to c to make m? To make $1-y$?

3. If a pounds of sugar cost b cents, what will x pounds cost? $x-y$ pounds?

4. What part of 9 is x? What part of x is y?

5. If 5 barrels of flour cost \$$a$, how many may be bought for \$$b$? For \$$\frac{3}{2}$?

6. How many oranges at a¢ each may be bought for b lemons at c¢ each?

7. Out of \$$x$ and y dimes, a boy spent a¢. How many cents did he have left? How many dimes? How many dollars?

8. A employs of B k of his men for a days. For how many days must 5 of A's men work for B to balance the account?

9. A number is composed of 3 digits, x, y, and z. What is the number? What if the digits are reversed in order?

10. If Q is one factor of S, what is the other?

11. Write 5 consecutive numbers, of which n is the greatest; the least; the middle one.

12. What is the next odd number above $2n$? Next even number?

13. What is the next odd number below $2n$? Next even number?

14. Answer questions 12 and 13 for the number $2n+1$; also for $2n-1$.

15. C men can dig a ditch in k days. How long would it take one man to dig it at the same rate?

16. How far can a bicyclist go in 37 minutes, if he can go x miles in c hours?

17. The sum of three numbers is x. A is one of them. The other two are alike. What are the numbers?

163. **A Statement of a Problem** is the one or more equations expressing the conditions of the problem.

NOTE. Before a problem can be solved it must be changed into algebraic symbols. The sentences of the problem must be changed into algebraic sentences called equations.

The student should

(*a*) Read the problem carefully;
(*b*) Notice what quantity is to be found, and represent it by x;
(*c*) See what quantities are given, and their relations to one another and to the unknown quantity;
(*d*) Consider only one relation at a time;
(*e*) Write each relation as soon as discovered; and
(*f*) Place these relations in the form of an equation.

Model Solution No. 1

A's age is double B's, B's is triple C's, and the sum of their ages is 140. What is the age of each?

Let $x =$ C's age.

Then $3x =$ B's, and $6x =$ A's.

1. $6x + 3x + x = 140$, the statement.
2. $10x = 140$, by collecting terms.
3. $x = 14$, C's age.
4. $3x = 42$, B's age.
5. $6x = 84$, A's age.

Verification

$$84 + 42 + 14 = 140;$$
$$140 \equiv 140.$$

Model Solution No. 2

A's age is R times B's, B's is S times C's, and the sum of their ages is T. Required their ages.

Let $x =$ C's.

Then $Sx =$ B's, and $RSx =$ A's.

1. $RSx + Sx + x = T$, the statement.
2. $(RS + S + 1)x = T$, by collecting terms.
3. $x = \frac{T}{RS + S + 1}$, C's age.
4. $Sx = \frac{ST}{RS + S + 1}$, B's age.
5. $RSx = \frac{RST}{RS + S + 1}$, A's age.

Verification No. 1

$$\frac{RST}{RS+S+1}+\frac{ST}{RS+S+1}+\frac{T}{RS+S+1}=T,$$

by substituting (3) in (1).

$$\frac{(RS+S+1)T}{RS+S+1}=T, \text{ by addition.}$$

$$T\equiv T, \text{ by division.}$$

Verification No. 2

Results obtained from literal equations may be verified by substituting a numerical value for each of the known quantities (letters) involved. For instance, in this example,

let $S=3,\ R=2,$ and $T=140.$

Then
$$x=\frac{140}{6+3+1}=\frac{140}{10}=14;$$

$$Sx=\frac{420}{6+3+1}=\frac{420}{10}=42;$$

$$RSx=\frac{840}{6+3+1}=\frac{840}{10}=84.$$

Substituting these values in (1), $84+42+14=140$, $140\equiv 140$.

MODEL SOLUTION No. 3

The sum of two numbers is 32, and their difference 6. What are the numbers?

Let $Q=$ the less number.

Then $Q+6=$ the larger number.

1. $Q+Q+6=32$, the statement.
2. $Q+Q=32-6$, by transposition.
3. $2Q=26$, by collecting terms.
4. $Q=13$, by division.
5. $Q+6=19$, by adding 6.

Verification

$13+13+6=32$, by substituting value of Q in (1).

$32\equiv 32.$

EXAMPLES

Solve and verify:

1. The sum of two numbers is h, and their difference is g. What are the numbers? Verify by substituting numerical values.

2. A man sold a horse and carriage for $200, $\frac{1}{2}$ the price of the horse equaling $\frac{1}{3}$ of the price of the carriage. What was the selling price of each?

3. A cistern can be filled by 3 pipes as follows: by the first in $2\frac{1}{3}$ hours, by the second in $3\frac{1}{2}$ hours, and by the third in 4 hours. In what time running together would they fill it?

4. The hind and fore wheels of a carriage have circumferences of 16 and 14 feet respectively. How far has it gone when the fore wheels have made 51 revolutions more than the others?

5. Four men enter into partnership with a capital of $4755. A furnished $\frac{1}{3}$ as much as B, C as much as A and B, and D as much as B and C. How much did each furnish?

6. The sum of $54 is divided among three men. John gets six times as much as Charles. Henry and Charles together have as much as John. How much does each get?

7. In a mile race the uniform speeds of two runners are proportional as 11 to 8. The poorer runner has 320 yards start, and is beaten by 30 seconds. What is the speed per second of each runner?

8. A student out for exercise walks a certain distance at the rate of 3 miles an hour; he then turns back, and running at the rate of 10 miles an hour reaches home again in an hour and three quarters. Required the total distance traveled.

9. A bicyclist going at the rate of 12 miles an hour is 20 minutes behind time when he has yet 25 miles to go. How much must he increase his speed to reach his destination in time?

10. A son earns 37 cents a day less than does his father, and in 8 days the father earns $6.08 more than the son earns in 5 days. What is the daily wages of each?

11. Two railroad trains, the first going at the rate of 20 miles an hour, meet and pass each other in 3 seconds. Had they been going in the same direction, the second train would have passed the first in 27 seconds. If the first train is 100 feet long, how long is the second?

Hints for analysis: 20 miles an hour are how many feet a second? If the second train is x feet long, what distance do the trains go in passing each other in 3 seconds? How far in 1 second? If this is the combined rate of speed, what is the rate of the second train? What is the gain in feet per second? Which train makes the gain? How long will it take to gain $100 + x$ feet?

12. A flock of sheep increases each year by $\frac{1}{4}$ of its number, and at the end of each year 16 are sold. After the 16 had been sold at the end of the third year, 189 were left. Required the number at first.

13. The sum of six consecutive numbers is 69. Find them.

14. Three trains run on a railroad between A and B, situated 200 miles apart. The trains run at the rate of 25, 20, and 30 miles an hour respectively. The first leaves A at 7 A.M., the second at 10.15 A.M., and the third leaves B at 11.30 A.M. Where will the third train be equidistant from the first and second.

Hints for analysis: If $x =$ the distance from B, how long will it take the third train to reach the point? What will be the time of day? How many hours has the first train traveled? The second? How far has the first train traveled? The second? The third? Reckoning from A, what expresses the difference in distance between the first and third? The third and second? Why are these differences equal?

15. A student walked from Ann Arbor to a certain village at the rate of 4 miles an hour. He decided to return by rail, and at the station was obliged to wait 10 minutes for a train which was

then 5 miles away. Arriving at his rooms, which were a mile from the Ann Arbor station, he found he had been gone $3\frac{1}{4}$ hours. Find the distance of the village from his rooms, assuming the distance traveled on the railroad and on the highway to be the same, and the rate of the student and the train to be uniform.

Hints for analysis: How long was the student in going to the village station? What was the train's rate? How long a time was required to reach Ann Arbor? How long to go from the Ann Arbor station to the student's rooms at his former rate of walking?

16. A can do a piece of work in 20 days, and B in 30 days. A begins the work, but stops, and B finishes it. B worked 10 days longer than A. How long did A work?

17. The width of a room is $\frac{7}{8}$ of its length. If the width had been 3 feet more, and the length 3 feet less, the room would have been square. Find its dimensions.

18. A and B can do a piece of work in a days; B and C in b days; C and D in c days; A and D in d days. In how many days can all working together do it?

19. A and B were ascending a tower, A being always 24 steps below B. When B was halfway up, he said to A, "When I shall have reached the top you will have ascended 8 times as high as you are now." How many steps had the tower?

20. A boy started from home on his bicycle at 8 A.M., going at the rate of $10\frac{1}{2}$ miles an hour. After riding a certain distance his wheel broke down, and he had to return afoot. If he reached home at 12.30 P.M. by walking at the rate of $3\frac{3}{4}$ miles per hour, how far in all did he travel?

21. Two clocks, A and B, are poorly regulated, A gaining 15 minutes and B 3 minutes an hour. A was set to the right time at 12 o'clock, when B was 21 minutes fast. At this moment B is $2\frac{3}{10}$ times as fast as A. What is the correct time now?

SYNOPSIS FOR REVIEW, CHAPTER III

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CHAPTER IV

SPECIAL THEOREMS AND SYMMETRY

THEOREMS AND DETACHED COEFFICIENTS

164. Theorem 1. The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.

Dem. Let $x + a$ be the sum of two quantities, then $(x + a)^2 = x^2 + 2\,xa + a^2$, by actual multiplication.

MENTAL EXERCISES

1. $(c + d)^2$.
2. $(2\,x + 3\,y)^2$.
3. $(a^2 + 2\,b)^2$.
4. $(1 + 3\,xy)^2$.
5. $(\frac{1}{2}\,x + \frac{2}{3}\,y)^2$.
6. $(\frac{3}{5}\,Q + \frac{5}{8}\,R)^2$.
7. $(2\frac{2}{3}\,m^2 + 3\frac{3}{4}\,n^3)^2$.
8. $\{(a + b) + c\}^2$.
9. $\{(a + b) + (c + d)\}^2$.
10. $\{2\,(e + f) + 3\,(g + h)\}^2$.

165. Theorem 2. The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.

Dem. Let $x - a$ be the difference of two quantities, then $(x - a)^2 = x^2 - 2\,xa + a^2$, by actual multiplication.

MENTAL EXERCISES

1. $(x-y)^2$.
2. $(3a-b)^2$.
3. $(b^2-z)^2$.
4. $(ab-1)^2$.
5. $(2a^2-3b^2)^2$.
6. $(x^m-a^n)^2$.
7. $(\frac{1}{2}c^2-\frac{2}{3}d^3)^2$.
8. $(\frac{3}{2}Q-\frac{4}{3}R)^2$.
9. $\{(a+b)-c\}^2$.
10. $\{(a-b)-(c-d)\}^2$.

166. Theorem 3. The product of the sum and difference of two quantities is equal to the square of the first minus the square of the second.

Dem. Let $x+a$ be the sum and $x-a$ the difference of two quantities, then $(x+a)(x-a)=x^2-a^2$, by actual multiplication.

Let the student perform the multiplication.

MENTAL EXERCISES

1. $(x+y)(x-y)$.
2. $(\frac{1}{2}y+2R)(\frac{1}{2}y-2R)$.
3. $(\frac{2}{3}x^2+\frac{3}{4}y^3)(\frac{2}{3}x^2-\frac{3}{4}y^3)$.
4. $(m^2+\frac{1}{7}n^2)(m^2-\frac{1}{7}n^2)$.
5. $(m+1-n)(1-m+n)$.
6. $\{(a+b)+c\}\{(a+b)-c\}$.
7. $\{x-(y-z)\}\{x+(y-z)\}$.
8. $(3a-5m)(3a+5m)$.
9. $(\frac{2}{3}Q+\frac{5}{7}S)(\frac{2}{3}Q-\frac{5}{7}S)$.
10. $(ab+c^2)(ab-c^2)$.
11. $(x^m+y^m)(x^m-y^m)$.
12. $(x^{2n}-y^{2n})(x^{2n}+y^{2n})$.
13. $(x-y-z)(x+y+z)$.
14. $(x^2-xy+y^2)(x^2+xy+y^2)$.
15. $(5^2a^3b^4+3^2xy^2z^3)(5^2a^3b^4-3^2xy^2z^3)$.
16. $(a+b+c+d)(a+b-c-d)$.
17. $(a-b-c-d)(a-b+c+d)$.
18. $(ax+1+by)(ax-1+by)$.
19. $(ax-by-1-cz)(ax-by+1+cz)$.

167. Theorem 4. The product of two binomials of the form $x+a$ and $x+b$ is equal to the square of the first term, plus the sum of the second terms multiplied by the first, plus the product of the second terms.

Dem. Let x, a, and b be any three quantities. Then $(x+a)(x+b)=x^2+(a+b)x+ab$, by actual multiplication.

Model Solutions

1. $(x+10)(x+5)=x^2+(10+5)x+50=x^2+15x+50.$
2. $(x-3)(x-8)=x^2+(-3-8)x+24=x^2-11x+24.$
3. $(x+3)(x-8)=x^2+(3-8)x-24=x^2-5x-24.$
4. $(x-6)(x+10)=x^2+(-6+10)x-60=x^2+4x-60.$

MENTAL EXERCISES

1. $(x+1)(x-5)$.
2. $(x-2)(x+7)$.
3. $(x+10)(x+8)$.
4. $(x-7)(x-9)$.
5. $(ax+5)(ax-4)$.
6. $(x^2-a^2)(x^2+b^2)$.
7. $(xy-ab)(xy+cd)$.
8. $(wx-3)(wx+a)$.
9. $(abcx-5)(abcx+6)$.
10. $(x^4y^4-3z)(x^4y^4-2z)$.
11. $(x+y+9)(x+y-6)$.
12. $(x-y-7)(x-y+5)$.
13. $(z+w-a)(z+w-b)$.
14. $(R+Q-S)(R+Q+T)$.
15. $(F+2G+5)(F+2G-3)$.
16. $(H-3I-7)(H-3I+2)$.

168. Theorem 5. The product of two binomials of the form $ax+b$ and $cx+d$ is equal to the product of the first terms of the binomials, plus the product of the second terms, plus the sum of the products of the extremes and the means.

Dem. Let $ax+b$ and $cx+d$ represent two binomials of the required form. Then $(ax+b)(cx+d)=acx^2+(ad+bc)x+bd$, by actual multiplication.

Model Solutions

1. $(2x+3)(3x+5) = 6x^2 + (10+9)x + 15 = 6x^2 + 19x + 15.$
2. $(2x-3)(3x-5) = 6x^2 + (-10-9)x + 15 = 6x^2 - 19x + 15.$
3. $(2x-3)(3x+5) = 6x^2 + (10-9)x - 15 = 6x^2 + x - 15.$
4. $(2x+3)(3x-5) = 6x^2 + (-10+9)x - 15 = 6x^2 - x - 15.$

MENTAL EXERCISES

1. $(x+5)(2x-3)$.
2. $(3x+4)(2x+5)$.
3. $(4x-3)(7x+2)$.
4. $(5x-3)(3x-8)$.
5. $(4xy+ab)(3xy-ab)$.
6. $(2c+d)(4c-5d)$.
7. $(9x+7y)(5x+y)$.
8. $(x-1)(2x+1)$.
9. $(3-5y)(7+6y)$.
10. $(9+2y)(3-5y)$.
11. $(y+3x)(3x+4y)$.
12. $(3abx-w)(3abx-3w)$.
13. $\{3(x+y)-5\}\{2(x+y)-7\}$.
14. $\{10(x-y)-3\}\{7(x-y)+4\}$.
15. $\{6(a-b)-5xy\}\{3(a-b)+8xy\}$.
16. $\{6(x+y)-b\}\{c(x+y)+d\}$.
17. $\{c(x-y)-9d\}\{m(x-y)+3d\}$.
18. $(3cdx-5ghy)(7cdx+4ghy)$.
19. $(2abx-3cdy)(4efx+5ghy)$.
20. $(5QR^2x-7ST)(3QR^2x+4ST)$.
21. $\{3(a+b+c)-5\}\{7(a+b+c)+6\}$.
22. $\{5(a+b)-3(c-d)\}\{2(a+b)+(c-d)\}$.

169. A **Continued Product** is the result obtained by multiplying together three or more factors.

170. Cor. The continued product of several algebraic expressions is the product of any two of them multiplied by a third, and so on until all the factors have been used.

MODEL SOLUTION

Find the continued product of $(x+a)(x+b)(x+c)$.

$$(x+a)(x+b) = x^2 + (a+b)x + ab, \text{ by Theorem 4.}$$

$$(x^2 + (a+b)x + ab)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc,$$

as shown by the following process of multiplication:

$$\begin{array}{l}
x^2 + (a+b)x + ab \\
x + c \\
\hline
x^3 + (a+b)x^2 + abx \\
\qquad\quad + cx^2 \quad + (ac+bc)x + abc \\
\hline
x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc
\end{array}$$

Let the student discover and frame a law by which such a product may be written out immediately.

MENTAL EXERCISES

1. $(x-a)(x+b)(x+c)$.
2. $(x+a)(x-b)(x-c)$.
3. $(x+1)(x+2)(x+3)$.
4. $(x+2)(x+3)(x+4)$.
5. $(x-2)(x-1)(x+3)$.
6. $(x-2)(x+3)(x-4)$.
7. $(a^2-b^2)^2(a^2+b^2)^2$.
8. $(ax+by)^2(ax-by)^2$.
9. $(a+b)(a-b)(a^2+b^2)$.
10. $(x+3)(x+3)(x-3)(x-3)$.
11. $(x+1)(x-1)(x^2+1)(x^4+1)(x^8+1)$.
12. $(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

171. Theorem 6. The square of any polynomial is equal to the sum of the squares of the several terms, plus the sum of twice the products of each term by each term that follows it. That is,

$$(a+b+c+d+\cdots+z)^2 = \Sigma a^2 + \Sigma\, 2\,ab.$$

Dem. Let m and n be any two quantities of any value. $m+n$ will represent their sum, and $(m+n)^2 = m^2 + n^2 + 2\,mn$

their square. Since m may have any value, substitute $a+b$ for it. Then

$$(a+b+n)^2=(a+b)^2+n^2+2(a+b)n$$
$$=a^2+b^2+n^2+2\,ab+2\,an+2\,bn.$$

Since n may have any value, substitute $c+x$ for it. Then

$$\begin{aligned}(a+b+c+x)^2\\ &=a^2+b^2+(c+x)^2+2\,ab+2\,a(c+x)+2\,b(c+x)\\ &=a^2+b^2+c^2+x^2+2\,cx+2\,ab+2\,ac+2\,ax+2\,bc+2\,bx\\ &=a^2+b^2+c^2+x^2+2\,ab+2\,ac+2\,ax+2\,bc+2\,bx+2\,cx.\end{aligned}$$

In like manner $d+y$ may be substituted for x, and so on indefinitely. Hence $(a+b+c+\cdots+z)^2=\Sigma a^2+\Sigma\,2\,ab$. Why?

Model Solution

$$\begin{aligned}(2\,x-3\,y^2-4\,z^3)^2 &=(2\,x)^2+(-3\,y^2)^2+(-4\,z^3)^2+2(2\,x)(-3\,y^2)\\ &\quad+2(2\,x)(-4\,z^3)+2(-3\,y^2)(-4\,z^3)\\ &=4\,x^2+9\,y^4+16\,z^6-12\,xy^2-16\,xz^3+24\,y^2z^3.\end{aligned}$$

MENTAL EXERCISES

1. $(a+b+c)^2$.
2. $(a-b+c)^2$.
3. $(a+2\,b-c)^2$.
4. $(x+y+R+Q)^2$.
5. $(3\,a^2+2\,b-c)^2$.
6. $(M-N+O-P)^2$.
7. $(1+2\,x-3\,y)^2-(3\,y^2+2\,x-1)^2$.
8. $(a^2-b^2-c^2-d^2)^2-(a^2+b^2+c^2+d^2)^2$.
9. $(x-y+1-m)^2-(x+m+y-1)^2$.
10. $(\frac{1}{2}\,x-2\,k+s)^2+(\frac{1}{3}\,x+k+s)^2$.
11. $(\frac{1}{3}\,x-\frac{1}{2}\,y+z)^2-(\frac{1}{3}\,x+\frac{1}{2}\,y-z)^2$.
12. $(\frac{1}{2}\,x+\frac{1}{3}\,y+\frac{1}{4}\,z)^2+(\frac{1}{2}\,x-\frac{1}{3}\,y-\frac{1}{4}\,z)^2$.
13. $(\frac{1}{2}\,a^2-\frac{2}{3}\,b^3-\frac{3}{4}\,c^4)^2-(a^2+b^3+c^4)^2$.
14. $(a+b+c+d+e+f+g+h)^2$.

172. Theorem 7. The cube of a binomial is equal to the cube of the first term, plus three times the square of the first term multiplied by the second, plus three times the first term multiplied by the square of the second, plus the cube of the second. That is,

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

Dem. 1. $(a+b)^2 = a^2 + 2\,ab + b^2$, by Theorem 1.

2. $(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$, by multiplying both members of (1) by $a+b$.

MODEL SOLUTION

$$(2\,x^2 - 3\,y^3)^3 = (2\,x^2)^3 + 3(2\,x^2)^2(-3\,y^3) + 3(2\,x^2)(-3\,y^3)^2 + (-3\,y^3)^3$$
$$= 8\,x^6 - 36\,x^4y^3 + 54\,x^2y^6 - 27\,y^9.$$

MENTAL EXERCISES

1. $(a+x)^3$.
2. $(a-b)^3$.
3. $(ax+y)^3$.
4. $(2\,x-y)^3$.
5. $(3\,x-2\,y)^3$.
6. $(5\,x+4\,y)^3$.
7. $(3\,x^2-2\,s^3)^3$.
8. $(\frac{1}{2}\,R-\frac{1}{8}\,Q)^3$.
9. $(\frac{2}{3}\,m-\frac{3}{4}\,n)^3$.
10. $(5\,T^2-\frac{2}{5}U^3)^3$.
11. $(6-4)^3$.
12. $(1+1)^3$.
13. $(1-1)^3$.
14. $(1+0)^3$.
15. $(\frac{1}{2}+\frac{1}{3})^3$.

173. Theorem 8. The cube of any polynomial is equal to the sum of such terms as a^3, plus the sum of such terms as $3\,a^2b$, plus the sum of such terms as $6\,abc$. That is,

$$(a+b+c+\cdots+z)^3 = \Sigma a^3 + \Sigma\, 3\,a^2b + \Sigma\, 6\,abc.$$

Dem. Let m and n be two quantities of any value. Then

1. $(m+n)^3 = m^3 + n^3 + 3\,m^2n + 3\,mn^2$, by Theorem 7.

Since m may have any value, substitute $a+b$ for it. Then

$$2.\quad (a+b+n)^3 = (a+b)^3 + n^3 + 3(a+b)^2n + 3(a+b)n^2$$
$$= a^3 + b^3 + n^3 + 3\,a^2b + 3\,ab^2 + 3\,a^2n + 3\,an^2$$
$$+ 3\,b^2n + 3\,bn^2 + 6\,abn.$$

Substituting $c+x$ for n in (2) and expanding,

$$3.\quad (a+b+c+x)^3 = a^3 + b^3 + c^3 + x^3 + 3\,a^2b + 3\,ab^2 + 3\,a^2c$$
$$+ 3\,ac^2 + 3\,a^2x + 3\,ax^2 + 3\,b^2c + 3\,bc^2 + 3\,b^2x + 3\,bx^2$$
$$+ 3\,c^2x + 3\,cx^2 + 6\,abc + 6\,abx + 6\,acx + 6\,bcx.$$

By substituting $d+y$ for x, and so on, this process may be continued. Only three kinds of terms are discovered, *i.e.*, Σa^3, $\Sigma\, 3\,a^2b$, $\Sigma\, 6\,abc$.

$$\therefore\ (a+b+c+d+\cdots+z)^3 = \Sigma a^3 + \Sigma\, 3\,a^2b + \Sigma\, 6\,abc.$$

MODEL SOLUTION

$$(2\,a^3+3\,b^2-c)^3 = (2\,a^3)^3 + (3\,b^2)^3 + (-c)^3 + 3(2\,a^3)^2(3\,b^2) + 3(2\,a^3)(3\,b^2)^2$$
$$+3(2\,a^3)^2(-c) + 3(2\,a^3)(-c)^2 + 3(3\,b^2)^2(-c)$$
$$+3(3\,b^2)(-c)^2 + 6(2\,a^3)(3\,b^2)(-c)$$
$$= 8\,a^9 + 27\,b^6 - c^3 + 36\,a^6b^2 + 54\,a^3b^4 - 12\,a^6c + 6\,a^3c^2 - 27\,b^4c$$
$$+ 9\,b^2c^2 - 36\,a^3b^2c.$$

MENTAL EXERCISES

1. $(a+b-c)^3$.
2. $(x-y+z)^3$.
3. $(P-Q-R)^3$.
4. $(2\,a+3\,b+1)^3$.
5. $(2\,a-3+c)^3$.
6. $(a^2-2\,b-3\,c^3)^3$.
7. $(\frac{1}{2}\,a+\frac{1}{3}\,b+\frac{1}{4}\,c)^3$.
8. $(4+2-3)^3$.
9. $(a+b+c+d)^3$.
10. $(a+0+c+0)^3$.
11. $(1+0-1+1)^3$.
12. $(a-b+c-d)^3$.

Multiplication by Detached Coefficients

174. When the letters of both multiplicand and multiplier can be arranged according to the same law, this law will hold good in the product. The multiplication can be effected by **using the coefficients alone** at first, and afterward writing the literal factors in the product according to the observed law.

Model Solution No. 1

Multiply $2a^3-3a^2x+5ax^2-x^3$ by $2a^2-ax+7x^2$.

METHOD OF COEFFICIENTS

$$\begin{array}{r}
2-3+5-1 \\
2-1+7 \\
\hline
4-6+10-2 \\
-2+3-5+1 \\
+14-21+35-7 \\
\hline
4-8+27-28+36-7
\end{array}$$

ORDINARY METHOD

$$\begin{array}{llllll}
2a^3 & -3a^2x & +5ax^2 & -x^3 & & \\
2a^2 & -ax & +7x^2 & & & \\
\hline
4a^5 & -6a^4x & +10a^3x^2 & -2a^2x^3 & & \\
 & -2a^4x & +3a^3x^2 & -5a^2x^3 & +ax^4 & \\
 & & +14a^3x^2 & -21a^2x^3 & +35ax^4 & -7x^5 \\
\hline
4a^5 & -8a^4x & +27a^3x^2 & -28a^2x^3 & +36ax^4 & -7x^5
\end{array}$$

Model Solution No. 2

Multiply x^3+2x-4 by x^2-1.

$$\begin{array}{r}
1+0+2-4 \\
1+0-1 \\
\hline
1+0+2-4 \\
-1-0-2+4 \\
\hline
1+0+1-4-2+4.
\end{array}$$

The exponent of the first term of the product is found by adding the exponents of the first terms in the multiplicand and multiplier. The exponents decrease by 1 to the right.

$\therefore\ x^5+0x^4+x^3-4x^2-2x+4=$ Product.

EXAMPLES

Multiply by detached coefficients:

1. $3a+2b$ by $2a-3b$.
2. $a^2-2ab+b^2$ by $a^2+2ab+b^2$.
3. $3a^2+4ax-5x^2$ by $2a^2-6ax+4x^2$.

4. $(a^4 - z^4)(a + z)$.
5. $(m^3 + n^3)(m^3 - n^3)$.
6. $a^3 + a^2x + ax^2 + x^3$ by $a - x$.
7. $2a^3 - 3ab^2 + 5b^3$ by $2a^2 - 5b^2$.
8. $x^3 - 3x^2 + 3x - 1$ by $x^2 - 2x + 1$.
9. $(x^2 + xy + y^2)(x^2 - xy + y^2)(x + y)(x - y)$.

Division by Detached Coefficients

175. Division by detached coefficients may be used in all cases where multiplication by detached coefficients can be used, as shown by the following:

Model Solution No. 1

Divide $10a^4 - 27a^3x + 34a^2x^2 - 18ax^3 - 8x^4$ by $2a^2 - 3ax + 4x^2$.

The law discovered in the dividend and divisor is that the exponents of a decrease by 1 to the right, and the exponents of x begin with unity in the second term and increase by 1 to the right.

METHOD OF DETACHED COEFFICIENTS

$$\begin{array}{rrrrr|l}
10 & -27 & +34 & -18 & -8 & 2-3+4 \\ \cline{6-6}
10 & -15 & +20 & & & 5-6-2 \\ \cline{1-3}
 & -12 & +14 & -18 & & \\
 & -12 & +18 & -24 & & \\ \cline{2-4}
 & & -4 & +6 & -8 & \\
 & & -4 & +6 & -8 & \\ \cline{3-5}
\end{array}$$

Quotient $= 5a^2 - 6ax - 2x^2$.

ORDINARY METHOD

$$\begin{array}{rrrrr|l}
10a^4 & -27a^3x & +34a^2x^2 & -18ax^3 & -8x^4 & 2a^2 - 3ax + 4x^2 \\ \cline{6-6}
10a^4 & -15a^3x & +20a^2x^2 & & & 5a^2 - 6ax - 2x^2 = \text{Quotient.} \\ \cline{1-3}
 & -12a^3x & +14a^2x^2 & -18ax^3 & & \\
 & -12a^3x & +18a^2x^2 & -24ax^3 & & \\ \cline{2-4}
 & & -4a^2x^2 & +6ax^3 & -8x^4 & \\
 & & -4a^2x^2 & +6ax^3 & -8x^4 & \\ \cline{3-5}
\end{array}$$

Model Solution No. 2

Divide a^3+b^3 by $a+b$.

$$\begin{array}{l|l}
1+0+0+1 & 1+1 \\
\underline{1+1} & \overline{1-1+1} \\
\quad -1+0 & \text{Quotient} = a^2-ab+b^2. \\
\quad \underline{-1-1} & \\
\qquad +1+1 & \\
\qquad \underline{+1+1} &
\end{array}$$

Model Solution No. 3

Divide h^6+k^6 by h^2+k^2.

$$\begin{array}{l|l}
1+0+0+0+0+0+1 & 1+0+1 \\
\underline{1+0+1} & \overline{1+0-1+0+1} \\
\quad +0-1+0 & \text{Quotient} = h^4+0\,h^3k-h^2k^2 \\
\quad \underline{+0+0+0} & +0\,hk^3+k^4 \text{ or } h^4-h^2k^2+k^4. \\
\qquad -1+0+0 & \\
\qquad \underline{-1-0-1} & \\
\qquad\quad +0+1+0 & \\
\qquad\quad \underline{+0+0+0} & \\
\qquad\qquad +1+0+1 & \\
\qquad\qquad \underline{+1+0+1} &
\end{array}$$

EXAMPLES

Divide:

1. x^3-1 by $x-1$.
2. x^4-y^4 by $x-y$.
3. $6\,a^4-96$ by $3\,a-6$.
4. x^7+y^7 by $x+y$.
5. x^4-y^4 by x^2-y^2.
6. x^5+a^5 by $x+a$.

176. Synthetic Division. When division by detached coefficients is practicable, as in the last article, the operation of division may be much condensed by performing mentally the operations of multiplying the divisor by the quotient, and subtracting the result from the dividend, and writing only the successive remainders. Thus, model No. 1 of the last preceding article may be worked as follows:

MODEL SOLUTION

$$\begin{array}{r|l} 10-27+34-18-8 & 2-3+4 \\ \cline{2-2} -12+14-18-8 & 5-6-\ 2 = \text{Quotient.} \\ -\ 4+\ 6-8 & \\ \hline 0 = \text{Remainder.} & \end{array}$$

Explanation. Dividing 10 by $2 = 5$, the first term of the quotient. Multiplying all the divisor by 5, and subtracting from the dividend, the first remainder is $-12+14-18-8$. $-12 \div 2 = -6$, the second term of the quotient. Multiplying all the divisor by -6, and subtracting from the new dividend, the second remainder is $-4+6-8$. $-4 \div 2 = -2$, the third term of the quotient. Multiplying and subtracting as before, the remainder is 0, and the quotient is $5a^2 - 6ax - 2x^2$.

EXAMPLES

Divide:

1. $x^7 - y^7$ by $x - y$.
2. $a^6 + b^6$ by $a^2 + b^2$.
3. $x^6 - y^6$ by $x^2 + xy + y^2$.
4. $x^4 + x^2a^2 + a^4$ by $x^2 - xa + a^2$.
5. $x^4 - 6x^2 - 16x + 21$ by $x - 3$.
6. $x^4 + 2x^3 - 7x^2 - 20x - 12$ by $x - 3$.
7. $y^6 - 6y^5 + 15y^4 - 20y^3 + 15y^2 - 6y + 1$ by $y^2 - 2y + 1$.

177. Binomial Divisor. The case of a binomial divisor of the first degree is of special importance, as it is with such divisor that the method of synthetic division is of chief value.

MODEL SOLUTION NO. 1

Divide $3x^3 - 14x^2 + 21x - 10$ by $x - 2$.

$$\begin{array}{l} 3 - 14 + 21 - 10 \;\underline{\lfloor -2} \\ \quad - \;\; 6 + 16 - 10 \\ \hline 3 - \;\; 8 + \;\; 5. \quad \therefore 3x^2 - 8x + 5 = \text{Quotient.} \end{array}$$

Explanation. Write the detached coefficients as above, with -2 at the right in the form of a divisor. Bring down the first coefficient, 3, multiply it by -2, write the result under the second, and subtract, giving -8. Multiply this remainder by -2, write the product under the third coefficient, 21, and subtract, giving $+5$. Multiply this remainder by -2, and subtract, giving no remainder. Hence, $x - 2$ is an exact divisor, and $3x^2 - 8x + 5$ is the quotient. If continued division is desired, the process of division may be repeated upon each succeeding quotient, as shown in the following example:

MODEL SOLUTION NO. 2

Divide

$x^4 - 45x^2 - 40x + 84$ by the factors $(x-1)(x+2)(x+6)(x-7)$.

$$\begin{array}{rl} & 1 + 0 - 45 - 40 + 84 \;\underline{\lfloor -1} \\ & \quad -1 - \;\; 1 + 44 + 84 \\ \hline \text{1st Quotient} = & 1 + 1 - 44 - 84 \;\underline{\lfloor +2} \\ & \quad +2 - \;\; 2 - 84 \\ \hline \text{2d Quotient} = & 1 - 1 - 42 \;\underline{\lfloor +6} \\ & \quad +6 - 42 \\ \hline \text{3d Quotient} = & 1 - 7 \;\underline{\lfloor -7} \\ & \quad -7 \\ \hline & 1 = \text{final Quotient.} \end{array}$$

Model Solution No. 3

Divide

$$x^3 - \tfrac{13}{12}x^2 + \tfrac{3}{8}x - \tfrac{1}{24} \text{ by the factors } (x - \tfrac{1}{2})(x - \tfrac{1}{3}).$$

$$\begin{array}{l}
1 - \tfrac{13}{12} + \tfrac{3}{8} - \tfrac{1}{24} \;\underline{\big|\, -\tfrac{1}{2}}\\
\underline{\quad -\tfrac{1}{2} + \tfrac{7}{24} - \tfrac{1}{24}}\\
1 - \tfrac{7}{12} + \tfrac{1}{12} \;\underline{\big|\, -\tfrac{1}{3}}\\
\underline{\quad -\tfrac{1}{3} + \tfrac{1}{12}}\\
1 - \tfrac{1}{4}. \quad \therefore\ x - \tfrac{1}{4} = \text{Quotient.}
\end{array}$$

Model Solution No. 4

Divide $x^5 + a^5$ by $x + a$.

$$\begin{array}{l}
1 + 0 + 0 + 0 + 0 + 1 \;\underline{\big|\, +1}\\
\underline{\quad +1 - 1 + 1 - 1 + 1}\\
1 - 1 + 1 - 1 + 1. \quad \therefore\ x^4 - x^3a + x^2a^2 - xa^3 + a^4 = \text{Quotient.}
\end{array}$$

Model Solution No. 5

Divide $x^5 + y^5$ by $x - y$.

$$\begin{array}{l}
1 + 0 + 0 + 0 + 0 \;+ 1 \;\underline{\big|\, -1}\\
\underline{\quad -1 - 1 - 1 - 1 \;- 1}\\
1 + 1 + 1 + 1 + 1, + 2
\end{array}$$

$x^4 + x^3y + x^2y^2 + xy^3 + y^4 =$ integral Quotient.

$+ 2\,y^5 =$ Remainder.

Model Solution No. 6

Divide $x^5 - 2\,x^4 - 15\,x^3 + 8\,x^2 + 68\,x + 48$ by $x - 2$.

$$\begin{array}{l}
1 - 2 - 15 + \;\; 8 + 68 \;+ 48 \;\underline{\big|\, -2}\\
\underline{\quad -2 - \;\; 0 + 30 + 44 \;- 48}\\
1 + 0 - 15 - 22 + 24, + 96
\end{array}$$

$x^4 + 0\,x^3 - 15\,x^2 - 22\,x + 24 =$ integral Quotient.

$+ 96 =$ Remainder.

178. It will be observed that this method of division gives not only the quotient, but also the **value of the remainder.** In models 1, 2, 3, and 4, the remainders are 0; therefore the divisors are exact divisors. In models 5 and 6, the remainders are $2y^5$ and 96; hence the divisors are not exact divisors. It may be observed also that this is a convenient method of finding the value of an integral expression in x for any particular value of x. For example, to find the value of the dividend in model No. 6, when $x=2$:

$$\begin{aligned} & x^5-2x^4-15x^3+8x^2+68x+48 \\ &= 2^5-2\times 2^4-15\times 2^3+8\times 2^2+68\times 2+48 \\ &= 32-32-120+32+136+48 \\ &= 96. \end{aligned}$$

But 96 is also the value of the remainder when the same expression is divided by $x-2$. Hence, in dividing by $x-2$, the remainder is equal to the result obtained by putting $+2$ in place of x. From this fact a Theorem of great importance is deduced, namely:

179. Theorem 9. *Remainder Theorem.* If any integral expression in x be divided by $x-a$, the remainder is equal to the result obtained by putting $+a$ in the place of x in the expression.

Dem. Let $Ax^n+Bx^{n-1}+Cx^{n-2}+\cdots+L$ be any integral expression in x. If it be divided by $x-a$, the remainder is equal to the result obtained by putting $+a$ in place of x in the expression. For, suppose upon trial $x-a$ is contained in the expression Q times with a remainder R, in which Q represents any series of terms which may arise from such division, and R any remainder not containing x. Now, since

the dividend equals the divisor multiplied by the quotient, plus the remainder, then

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + L = (x-a)Q + R.$$

But since R does not contain x, any change in the value of x will not produce a change in R. As x changes in value and approaches that of a, the value of $(x-a)Q$ becomes smaller and smaller, and in the limit, when x becomes practically equal to a, is zero. $\therefore$ putting a in place of x,

$$Aa^n + Ba^{n-1} + \cdots + L = (a-a)Q + R = R.$$

Why should not the divisor $x - a$ be considered zero? See note to Axiom (a) in Equations, Art. **150**.

180. Theorem 10. *Factor Theorem.* If any integral and rational expression in x vanishes when a is put in place of x, then $x - a$ is a factor of the expression.

Dem. For $Aa^n + Ba^{n-1} + Ca^{n-2} + \cdots + L = 0$ when $R = 0$. But if $R = 0$, the expression $Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + L$ is exactly divisible by $x - a$.

ILLUSTRATION. $x^{17} - a^{17}$ is divisible by $x - a$,

$$\because\ (+a)^{17} - a^{17} = 0 = R.$$

181. COR. 1. Putting $-a$ in place of x gives the remainder when the divisor is $x + a$.

Dem. For $x + a \equiv x - (-a)$, the form given in the Theorem.

ILLUSTRATION. $x + a$ is a factor of $x^{13} + a^{13}$,

$$\because\ (-a)^{13} + a^{13} = 0 = R.$$

182. COR. 2. Putting $\mp \frac{d}{c}$ in place of x gives the remainder when the divisor is $cx \pm d$.

Dem. This is the general case, and can be reduced to the form of $x \pm \frac{d}{c}$, which is equivalent to the form $x \pm a$.

ILLUSTRATION. To show that the general case may be brought under the case of $x - a$, take the following example:

Divide $3x^3 - 14x^2 + 21x - 10$ by $3x - 5$.

$$\frac{3x^3-14x^2+21x-10}{3x-5} = \frac{3x^3-14x^2+21x-10}{3(x-\frac{5}{3})} = \frac{1}{3} \cdot \frac{3x^3-14x^2+21x-10}{x-\frac{5}{3}}$$

$$= \tfrac{1}{3}(3x^2-9x+6) = x^2-3x+2, \text{ Quotient.}$$

METHOD OF SYNTHETIC DIVISION

$$\begin{array}{rrrrl}
3x^3 & -14x^2 & +21x & -10 & \underline{\lvert -\frac{5}{3}} \\
 & -5 & +15 & -10 & \\
\hline
3)3 & -9 & +6 & & = \text{Quotient 3 times too large. Why?} \\
\hline
1 & -3 & +2. & & \therefore x^2 - 3x + 2 = \text{Quotient.}
\end{array}$$

Hence putting $+\frac{5}{3}$ in place of x gives the remainder (in this case 0) when the divisor is $3x - 5$.

ILLUSTRATIONS. Is $16x^4 - 81y^4$ divisible by $2x \pm 3y$? Yes. $\because$ $16(-\frac{3}{2}y)^4 - 81y^4 = 0$, and $16(+\frac{3}{2}y)^4 - 81y^4 = 0$.

183. Theorem 11. *Division Theorem.* 1. The sum of two numbers is a divisor of the sum of their same odd powers, and of the difference of their same even powers. 2. The difference of two numbers is a divisor of the difference of the same powers of those numbers.

Dem. Each of these statements may be proved by the Remainder Theorem, as shown by the following diagram:

	DIVIDEND	DIVISOR	REMAINDER TEST	REMAINDER
1.	$x^n + a^n$	$x + a$	$(-a)^n + a^n =$	0 if n is odd; $2a^n$ if n is even.
2.	$x^n - a^n$	$x + a$	$(-a)^n - a^n =$	0 if n is even; $-2a^n$ if n is odd.
3.	$x^n - a^n$	$x - a$	$(+a)^n - a^n =$	0 always.
4.	$x^n + a^n$	$x - a$	$(+a)^n + a^n =$	$2a^n$ always.

Hence, $x^n + a^n$ is exactly divisible by $x + a$ if n is odd, but never by $x - a$; and $x^n - a^n$ is exactly divisible by $x - a$, and by $x + a$ if n is even.

184. Discovery of the laws for writing any quotient involved in Theorem 11.

1. Divide $x^n \pm a^n$ by $x + a$.

By synthetic division

$$\begin{array}{l} 1+0+0+0+0+\cdots\cdots\cdots\pm a^n \underline{|+a} \\ \underline{\quad +a-a^2+a^3-a^4\cdots\mp a^{n-1}\pm a^n} \\ 1-a+a^2-a^3+a^4\cdots\pm a^{n-1};\ R=0,\ -2a^n \text{ or } +2a^n. \end{array}$$

Integral Quotient $= x^{n-1} - x^{n-2}a + x^{n-3}a^2 - x^{n-4}a^3 + \cdots \pm a^{n-1}$.

Remainder $= 0$ when a^n is $+$ and n is *odd*, or a^n is $-$ and n is even; $2a^n$ when a^n is $+$ and n is *even*; $-2a^n$ when a^n is $-$ and n is odd.

2. Divide $x^n \pm a^n$ by $x - a$.

By synthetic division
$$\begin{array}{l} 1+0+0+0+\cdots\cdots\pm 1 \underline{|-1} \\ \underline{\quad -1-1-1+\cdots-1-1} \\ 1+1+1+1+\cdots+1;\ +2 \text{ or } 0 = R. \end{array}$$

Integral Quotient $= x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}$.

Remainder $= +2a^n$ when a^n in dividend is $+$; 0 when it is $-$.

185. From the above examples the **General Laws** for writing the quotient may be discovered, and stated as follows:

1. When the divisor is $x + a$, the signs in the quotient alternate, the first term being positive; and when the divisor is $x - a$, the signs in the quotient are all positive.

2. The exponent of x decreases by the exponent of x in the divisor, becoming 0 in the last term of the quotient.

3. a begins in the second term of the quotient, its exponent increasing by its exponent in the divisor until it becomes equal to the exponent of x in the first term.

4. The sum of the exponents of x and a in each term is always the same as the exponent of the first term.

NOTE. The student should notice carefully the form of the quotient in each of the above cases, and be able to write it without dividing. He should be able to tell at a glance also, whether such division is exact or not, and if not exact, to tell what the remainder, or fractional term of the quotient is. The following examples will serve for practice (see diagram, Theorem 11):

MENTAL EXERCISES

1. $(a^3 + 1) \div (a + 1)$.
2. $(a^3 + 1) \div (a - 1)$.
3. $(a^3 - 1) \div (a + 1)$.
4. $(a^3 - 1) \div (a - 1)$.
5. $(a^4 + 1) \div (a + 1)$.
6. $(a^4 + 1) \div (a - 1)$.
7. $(a^4 - 1) \div (a + 1)$.
8. $(a^4 - 1) \div (a - 1)$.
9. $(h^5 + k^5) \div (h + k)$.
10. $(h^5 - k^5) \div (h - k)$.
11. $(h^5 + k^5) \div (h - k)$.
12. $(h^5 - k^5) \div (h + k)$.
13. $(m^6 + n^6) \div (m + n)$.
14. $(m^6 + y^6) \div (m - y)$.
15. $(x^6 + y^6) \div (x^2 + y^2)$.
16. $(n^6 + p^6) \div (n^2 - p^2)$.
17. $(c^7 + d^7) \div (c + d)$.
18. $(c^7 - d^7) \div (c - d)$.
19. $(x^6 - y^6) \div (x - y)$.
20. $(x^6 - y^6) \div (x + y)$.
21. $(x^6 - y^6) \div (x^2 + y^2)$.
22. $(x^6 - y^6) \div (x^2 - y^2)$.
23. $(x^6 - y^6) \div (x^3 - y^3)$.
24. $(x^6 - y^6) \div (x^3 + y^3)$.
25. $(x^8 + a^8) \div (x + a)$.
26. $(f^9 - g^9) \div (f - g)$.
27. $(l^{10} + m^{10}) \div (m^2 + l^2)$.
28. $(R^{12} + S^{12}) \div (R^4 + S^4)$.
29. $(R^{12} - S^{12}) \div (R^3 - S^3)$.
30. $(a^3b^6 + c^9) \div (ab^2 + c^3)$.
31. $(1 - v^5) \div (-v + 1)$.
32. $(27a^3 - 1) \div (3a - 1)$.
33. $(32a^5 - 243) \div (2a - 3)$.
34. $(a^{2s+1} + b^{2s+1}) \div (a + b)$.
35. $(a^{2s} - b^{2s}) \div (a + b)$.
36. $(a^{2s} - b^{2s}) \div (a - b)$.

SYMMETRY

186. An expression is **Symmetrical** with respect to any *two* quantities when its value is unaltered by any interchange of the two quantities.

ILLUSTRATIONS. $x+a$, x^2+xa+a^2, and $x^3+x^2a+xa^2+a^3$ are symmetrical with respect to x and a, because, substituting x for a, and a for x,

$$x+a \equiv a+x, \text{ by Law of Order.}$$

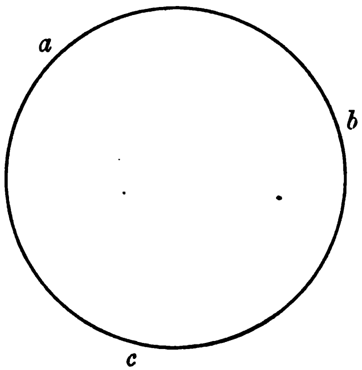
$$x^2+xa+a^2 \equiv a^2+ax+x^2, \text{ by Law of Order.}$$

$$x^3+x^2a+xa^2+a^3 \equiv a^3+a^2x+ax^2+x^3, \text{ by Law of Order.}$$

Is $a^3+b^3+c^3-3\,abc$ symmetrical with respect to a and b; to a and c; to b and c? Prove each case by substitution.

187. Three letters a, b, c are said to be in **Cyclic Order** when arranged as follows: *abc*, *bca*, *cab*, *abc*.

ILLUSTRATION. If the letters a, b, c are placed upon the circumference of a circle in the order named and the direction indicated by that order is constantly followed, they will be in groups *abc*, *bca*, *cab*, in which b takes the place of a, c of b, and a of c, as here shown.



188. An expression is **symmetrical** with respect to **three** of its quantities a, b, c if its value is unchanged when b is put for a, c for b, and a for c.

ILLUSTRATIONS. $a^2b+b^2c+c^2a$ is symmetrical with respect to the cycle *abc*. For changing a into b, b into c, c into a,

$a^2b+b^2c+c^2a \equiv b^2c+c^2a+a^2b$, which is evidently true by the Law of Order of terms. Likewise $(a-b)^2+(b-c)^2+(c-a)^2$ is symmetrical with respect to the cycle abc for it is identical with $(b-c)^2+(c-a)^2+(a-b)^2$ by Law of Order of terms.

189. In general, an expression is **symmetrical** with respect to the cycle $abc \cdots yz$ if it remains the same by putting b for a, c for b, $\cdots z$ for y, and a for z. Such a series of changes is called **Cyclic Displacement**, and the partial symmetry, as represented in $a^2b+b^2c+c^2a$, is called **Cyclo-symmetry**.

190. An expression is **Completely Symmetrical** with respect to three or more of its quantities when it is symmetrical with respect to each and every pair of them.

ILLUSTRATIONS. $x^3+y^3+z^3-3xyz$ is completely symmetrical with respect to x, y, and z, for it remains unchanged in value by interchanging x and y, or y and z, or z and x. So also is $3a^2b+3ab^2+3a^2c+3ac^2+3b^2c+3bc^2$, because any pair of letters may be interchanged without altering the value of the expression.

191. An expression which is *completely symmetrical* with respect to a set of quantities is also *cyclo-symmetrical* with respect to those quantities; but an expression which is cyclo-symmetrical is not necessarily completely symmetrical.

ILLUSTRATIONS. $a^3+b^3+c^3-3abc$ is both completely symmetrical and cyclo-symmetrical; but $a^2b+b^2c+c^2a$ is only cyclo-symmetrical, for when the letters of the pair a and b are interchanged, $a^2b+b^2c+c^2a \not\equiv b^2a+a^2c+c^2b$.

Symmetrical groups are frequently indicated by Σ, as has already been shown. Thus, $a+b+c+\cdots+z \equiv \Sigma a$; $a^2b+ab^2+a^2c+ac^2+\cdots+y^2z+yz^2 \equiv \Sigma a^2b$.

EXAMPLES

With respect to what letters are the following expressions symmetrical? Work out the steps in each proof.

1. $abc + 5ab + 2(a^3 + b^3)$.
2. $x^3 - y^3 - z^3 - 3xyz$.
3. $x^3 + y^3 - z^3 + 3xyz$.
4. $a^2 + a^2y + ay^2 + y^2$.
5. $(a-b)^4$.
6. $x^3 - y^3 + z^3 + 3xyz$.
7. $(a-b)^4 + (a+b)^4$.
8. $x^3 + y^3 + z^3 - 3xyz$.
9. $yz(y-z) - zx(x-z) - xy(y-x)$.
10. $x^3(y-z) + y^3(z-x) + z^3(x-y)$.
11. $x^3 + y^3 + z^3 + 3(x+y)(y+z)(z+x)$.
12. $(x-y)^2 + (y-z)^2 + (z-x)^2$.
13. $a(a+3b)^3 + b(b+3a)^3$.
14. $(x+y+z)(y^2+z^2+x^2-zx-xy-yz)$.

Write the following expressions by the use of Σ:

15. $a^2b + ab^2 + \cdots + mn^2 + \cdots + yz^2 + y^2z$.
16. $a^3 + b^3 + \cdots 3a^2b + 3ab^2 + \cdots + 6abc + 6abd + \cdots + 6xyz$.
17. $a^2(a-b) + b^2(b-c) + \cdots$.
18. $a^2bc + ab^2c + abc^2 + a^2bd + \cdots + xyz^2$.
19. $a^4 + b^4 + \cdots + 4a^3b + \cdots + 6a^2b^2 + \cdots + 12a^2bc + \cdots$
 $+ 24abcd + \cdots$.

Write in full with respect to a, b, c, and d:

20. Σa^2.
21. Σa^3.
22. $\Sigma(a-b)$.
23. $\Sigma ab(b-c)^2$.
24. Σa^2bc.
25. $\Sigma a^2(b+c)$.
26. $\Sigma a^2 + \Sigma 2ab$.
27. $\Sigma a^3 + \Sigma 3a^2b + \Sigma 6abc$.

Arrange the following expressions in cyclic order:

28. $(a-b)$, $(c-b)$, $-(a-c)$.
29. $(b-a)$, $-(a-c)$, $-(c-b)$.
30. $(a-b)(b-c)$, $-(c-a)(c-b)$, $(a-c)(b-a)$.

MISCELLANEOUS EXAMPLES

1. $(\frac{1}{2}ab^2+\frac{1}{8}a^2b)^2$.

2. $(\frac{3}{4}x^2-\frac{1}{2}x^3y)^2$.

3. $(ax-by+cz)^2$.

4. $(x-5)(x-8)$.

5. $(x-15)(x+7)$.

6. $(x+9)(x+10)$.

7. $(1-3x+3x^2-x^3)^2$.

8. $(\frac{2}{3}+7km)(\frac{2}{3}-7km)$.

9. $(x^6+y^6)\div(x^2+y^2)$.

10. $(x+y-7)(x+y+8)$.

11. $(m-n-o)(m-n+o)$.

12. $(a-2b)^3$.

13. $(4x^2-3y^3)^3$.

14. $(a^2-2ax+x^2)^3$.

15. $(2a-b+c-3d)^3$.

16. $(x^4+x^2+1)(x^2-1)$.

17. $(a+b)(c+d)(e-f)$.

18. $(x-k+m)(x+k-m)$.

19. $(x^2+x+1)(x^2-x+1)$.

20. $(81-16c^4)\div(3-2c)$.

21. $(5+a+b)(-7+a+b)$.

22. $(32x^5+243a^5)\div(2x+3a)$.

23. $(2x^2-3x-4)(2x^2-3x-4)$.

24. $(8c^2-6cd-5d^2)\div(4c-5d)$.

25. $(x^3+2x^2+3x+1)(x^3-2x^2+3x-1)$.

26. $\{(a-b)^2-(c-d)^2\}\{(a-b)^2+(c-d)^2\}$.

27. $(25x^4-36x^2y^2+4y^4)\div(5x^2-4xy-2y^2)$.

28. $(x^4-2ax^3+3x^2a^2-2a^3x-3a^4)\div(x^2-xa+3a^2)$.

29. Find the value of k in $x^3-13x^2+87x-k$ which will render the expression exactly divisible by $x-5$.

30. Is $x^3-8x^2+5x+14$ exactly divisible by $(x-7)(x+1)$?

31. If $x^3+0x^2-49x-120$ is exactly divisible by the expression $(x+3)(x+5)(ax-b)$, find the value of a and of b.

32. Is $x^3+a^3+b^3-3abx$ divisible by $x+a+b$. Use Factor Theorem.

33. $\{(3a+4b)^3+(4a+3b)^3\}\div(7a+7b)$. Use Theorem 11. $7a\equiv 3a+4a$.

SYNOPSIS FOR REVIEW, CHAPTER IV

SPECIAL THEOREMS AND SYMMETRY

- **Theorems and Detached Coefficients**

  Theorems:

  1. Square of sum of two quantities $(x+a)^2$, **164**. Dem.
  2. Square of difference of two quantities $(x-a)^2$, **165**. Dem.
  3. Product of sum and difference of two quantities $(x+a)(x-a)$, **166**. Dem.
  4. Product of two binomials, form of $(x+a)(x+b)$, **167**. Dem.
  5. Product of two binomials, form of $(ax+b)(cx+d)$, **168**. Dem.
     Continued Product, **169**.
     Corollary, **170**.
  6. Square of polynomial $(a+b+\cdots+z)^2$, **171**. Dem.
  7. Cube of binomial $(a+b)^3$, **172**. Dem.
  8. Cube of polynomial $(a+b+\cdots+z)^3$, **173**. Dem.
     Multiplication by Detached Coefficients, **174**.
     Division by Detached Coefficients, **175–185**.
     Synthetic Division, **176**. Binomial Divisor, **177**.
  9. Remainder Theorem, **179**. Dem.
  10. Factor Theorem, **180**. Dem.
      Cor. 1. Substituting $-a$ for x, **181**. Dem.
      Cor. 2. Substituting $\mp\frac{d}{c}$ for x, **182**. Dem.
  11. Division Theorem, **183**. Dem.
      Discovery of Laws under Theorem 11, **184**.
      General Laws for writing quotient, **185**.

- **Symmetry**

  1. Expression Symmetrical with respect to any two quantities, when, **186**.
  2. Cyclic Order, **187**.
  3. Expression Symmetrical with respect to three of its quantities, when, **188**.
  4. Cyclic Displacement, Cyclo-symmetry, **189**.
  5. Complete Symmetry, **190**.
  6. Relation of Complete Symmetry to Cyclo-symmetry, **191**.

CHAPTER V

FACTORING

SECTION I

FUNDAMENTAL PROPOSITIONS

192. A **Factor** is an exact divisor. The factors of a number are those numbers which multiplied together produce it.

193. A **Composite Number** is one composed of integral factors other than itself and unity.

194. A **Prime Number** is one containing no integral factors other than itself and unity.

195. Two or more numbers are **prime to each other** when they have no common integral factor other than unity.

ILLUSTRATION. 8 and 9 are composite numbers, but are prime to each other.

196. Prop. 1. A monomial can be resolved into its prime factors by (1) factoring the numerical coefficient, and (2) separating each of the literal quantities into as many factors as are indicated by the exponent of the letter.

Dem. See the definition of factor and of positive integral exponent.

Model Solution

Resolve $-24\,a^2m^3x^4$ into its prime factors.

$$-24 = -2 \cdot 2 \cdot 2 \cdot 3;$$
$$a^2 = a \cdot a;$$
$$m^3 = m \cdot m \cdot m;$$
$$x^4 = x \cdot x \cdot x \cdot x.$$
$$\therefore\ -24\,a^2m^3x^4 = -2 \cdot 2 \cdot 2 \cdot 3 \cdot a \cdot a \cdot m \cdot m \cdot m \cdot x \cdot x \cdot x \cdot x.$$

MENTAL EXERCISES

Factor:

1. $6\,a^3bc^5y^4$.
2. $-91\,m^2n^5x^3y$.
3. $1728\,x^4y^5z^3$.
4. x^{m+1}.
5. $1078\,xy^4z$.
6. $10911\,c^2d^3h^5g^4$.
7. $12\,x^{2m+2n}$.
8. $-45\,x^{2m+1}y^{n+3}z^{s+t}$.
9. $24\,a^2b^3(x-y)^2$.
10. $(a-b)^2(a+b)^3$.

197. Prop. 2. A factor which occurs in every term, or group of terms, of a polynomial is one of the factors of the polynomial; the other factor is the quotient of the polynomial divided by the common factor.

Dem. See Law of Distribution.

Model Solution

Factor $5\,ax - 10\,a^2xy + 30\,a^3x^2y^4z^5 + 5\,axz$.

$5\,ax$ is observed to be the common factor, because it is found in every term. Hence $5\,ax$ is one factor, and the quotient obtained by dividing the polynomial by $5\,ax$ is $1 - 2\,ay + 6\,a^2xy^4z^5 + z$, the other factor.

$$\therefore\ 5\,ax - 10\,a^2xy + 30\,a^3x^2y^4z^5 + 5\,axz = 5\,ax(1 - 2\,ay + 6\,a^2xy^4z^5 + z).$$

Note. Since this common factor may conceal some known form, every polynomial to be factored should be first tested by this proposition.

198. Cor. A polynomial which can be arranged in groups of two or more terms so that all the groups shall contain a common factor, can be factored by Prop. 2.

Model Solution

Factor $12\,x^3y^2 - 28\,xy^3 - 9\,a^2bx^2 + 21\,a^2by$.

$$12\,x^3y^2-28\,xy^3-9\,a^2bx^2+21\,a^2by=4\,xy^2(3\,x^2-7\,y)-3\,a^2b(3\,x^2-7\,y)$$
$$=(3\,x^2-7\,y)(4\,xy^2-3\,a^2b).$$

EXAMPLES

1. $27\,a^3x^2 - 18\,ax^2$.
2. $a - a^2 + a^3 - a^{n+1}$.
3. $126\,ax^2y^3 - 18\,a^2x^3y^5 + 24\,a^3x^2y^4$.
4. $a^{2m}b^{3m+1} - a^{3m+1}b^{2m}$.
5. $ac + ab - 2\,ax$.
6. $22\,x^2 - 110\,y^2 - 33\,xy$.
7. $as^3 - abs^2 + as$.
8. $70\,ab^2c - 105\,abc^2 + 35\,a^2bc$.
9. $108\,m^3n^6 + 54\,m^8n^4 - 243\,m^7n^8$.
10. $48\,g^2h - 16\,gh + 16\,g^2h^2 + 8\,g$.
11. $30\,r^3s^3t^2 + 20\,r^4s^4t^2 + 20\,r^2s^2t^2$.
12. $2\,k^5 + 12\,k^3f^2 + 8\,k^4f + 2\,kf^4 + 8\,k^2f^3$.
13. $7\,ab^2x - 28\,a^2bx^2 + 42\,a^3bx - 7\,abx$.
14. $4\,x^2z^3 - 7\,x^2z^4 + 12\,xz^5 - xz^3$.
15. $8\,a^2x^2y + 17\,amxy - 3\,a^2m^2x^2y$.
16. $32\,x^3 + 48\,x^2y + 28\,xy^2 - 4\,x$.
17. $26\,a^2c^3 - 39\,ac^4 + 65\,a^3c^5 - 91\,a^4c^6$.
18. $221\,v^2w^3 + 204\,v^5w^5q - 187\,q^4v^4w^7$.
19. $154\,xa^2cf^3 - 198\,x^2a^3c^2f^2 + 253\,f^2cag$.
20. $247\,pqrs + 209\,pq^2r^3s^4 - 285\,p^2q^4rs^5$.
21. $221\,deij - 255\,id^2je^4 + 272\,jdie^3 - 306\,e^2ijd$.
22. $411\,axym - 548\,xym - 685\,ym^2 + 822\,y^2m$.

23. $a(x-y)-b(x-y)$.

24. $\frac{1}{2}a^2-\frac{1}{3}a^4b+\frac{1}{4}a^5x$.

25. $a^{m+n}b^{m+n}-a^{n+s}b^{m+t}$.

26. $5^3-25+2.5^{n+2}-5^2\cdot a$.

27. $6x^2y-10x^3y^2-12x^2yz$.

28. $a^2nx-a^2mn-a^2xy+a^2my$.

29. $ab+a+b+1+bx+x$.

30. $a^2+2a^3+3+6a+2a+1$.

31. $x^2+ax-bx-ab+x+a$.

32. $x^5+6x^2-2x^4-3x^3+x-2$.

33. $acx^2+adx+bcx+bd$. Check.

34. $ap-cq+cs+aq-cp-as$.

35. $12x^5-8x^3y^2+21x^2y-14y^3$.

36. $64xy-10ax+96by-15ab$.

37. $2^s a^{n+2}b(x-y)^3-2^{2s}a^3b(x-y)^2$.

38. $2y-6ay-2x-4by+6ax+4bx$.

39. $21a-5b+3ab-35-14c-2bc$.

40. $2ax+3bx-cx-4ay-6by+2cy$.

41. $6sy+9ny-2px+6nx+4sx-3py$.

42. $45a^3x^3+30a^2bx^2y-27a^2cx^2y-18abcxy^2$.

43. $1070lu^2+1177u^3l^2+1391l^4u^2o^4-1605ulz$.

44. $2717xz-2223x^2-1729xy-5681x^2yz$.

199. Prop. 3. A trinomial, one of whose terms is plus (or minus) twice the product of the square roots of the other two, is equal to two binomial factors, each the sum (or difference) of these square roots.

Dem. See Theorems 1 and 2, Arts. **164** and **165**.

Model Solution

Factor $4x^2-12xy+9y^2$.

Since $-12xy$ is minus twice the product of the square roots of $4x^2$ and $9y^2$, the trinomial is equal to the two factors $(2x-3y)(2x-3y)$.

MENTAL EXERCISES

Factor:

1. $x^2 + 2xy + y^2$.
2. $a^2 - 2ab + b^2$.
3. $x^2 - 4x + 4$.
4. $x^2 + 18x + 81$.
5. $\frac{1}{4}s^2 - \frac{1}{3}s + \frac{1}{9}$.
6. $m^{2s} - m^s n^s + \frac{1}{4}n^{2s}$.
7. $16y^2 \pm 16yz + 4z^2$.
8. $16x^{2m-2} - 56x^{m-1} + 49$.
9. $9x^{4m} + 30x^{2m}y + 25y^2$.
10. $64 - 96 + 36$.
11. $\frac{4}{9}x^2 + 2xy + \frac{9}{4}y^2$.
12. $\frac{1}{4}g^4 + \frac{1}{2}g^2h + \frac{1}{4}h^2$.
13. $1 - 14kl + 49k^2l^2$.
14. $1 + 10Q^a + 25Q^{2a}$.
15. $36c^2 - \frac{12}{5}cd + \frac{1}{25}d^2$.
16. $a^2m^2x^4 - amx^2y^2 + \frac{1}{4}y^4$.
17. $(x+y)^2 + 2(x+y) + 1$.
18. $(g+h)^2 - 2f(g+h) + f^2$.
19. $9(c+d)^2 + 24(c+d) + 16$.
20. $81(R+S)^2 - 126(R^2 - S^2) + 49(R-S)^2$.

200. Cor. 1. A polynomial of the form $\Sigma a^2 + \Sigma 2ab = (a+b+c+\cdots)(a+b+c+\cdots)$, by Theorem 6.

201. Cor. 2. A polynomial of the form $\Sigma a^3 + \Sigma 3a^2b = (a+b)(a+b)(a+b)$, by Theorem 7.

202. Cor. 3. A polynomial of the form $\Sigma a^3 + \Sigma 3a^2b + \Sigma 6abc = (a+b+c+\cdots)(a+b+c+\cdots)(a+b+c+\cdots)$, by Theorem 8.

MENTAL EXERCISES

Factor:

1. $a^3 + b^3 + 3a^2b + 3ab^2$.
2. $a^3 - b^3 - 3a^2b + 3ab^2$.
3. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
4. $a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3a^2c + 3ac^2 + 3b^2c + 3bc^2 + 6abc$.
5. $a^2 + b^2 + c^2 + d^2 - 2ab - 2ac + 2ad + 2bc - 2bd - 2cd$.

6. $x^3 - 1 - 3x^2 + 3x$.

7. $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$.

8. $\Sigma a^3 + \Sigma 3a^2b + \Sigma 6abc$.

9. $8x^6 - 27y^9 - 36x^4y^3 + 54x^2y^6$.

10. $\frac{1}{4}x^2 + \frac{1}{9}y^2 + \frac{1}{16}z^2 - \frac{1}{3}xy + \frac{1}{4}xz - \frac{1}{6}yz$.

203. Prop. 4. The difference between two quantities is equal to two binomial factors which are: (1) the square root of the first quantity plus the square root of the second, and (2) the square root of the first quantity minus the square root of the second.

Dem. See Theorem 3.

Model Solution

$$9x^2 - 25y^2 = (3x + 5y)(3x - 5y).$$

204. Cor. 1. A polynomial which can be reduced to the difference of two squares can be factored by Prop. 4.

Model Solution

$$\begin{aligned} a^2 - 2ab + b^2 - c^2 - 2cd - d^2 &= (a-b)^2 - (c+d)^2 \\ &= \{(a-b)+(c+d)\}\{(a-b)-(c+d)\} \\ &= (a-b+c+d)(a-b-c-d). \end{aligned}$$

MENTAL EXERCISES

Factor:

1. $a^2 - 4b^2$.
2. $4x^6 - y^6$.
3. $25a^6 - 36b^4$.
4. $4a^2 - 9b^2$.
5. $x^{2m} - y^{2n}$.
6. $(x+y)^2 - z^2$.
7. $(x+y)^2 - (v-w)^2$.
8. $x^2 - 4$.
9. $a^6 - b^6$.
10. $x^4 - 16y^4$.
11. $x^8 - y^8$.
12. $x^{10} - y^{10}$.
13. $x^{24} - y^{24}$.
14. $x^{16} - y^{16}$.

15. $16x^6 - 4z^6$.

16. $9m^2 - 25n^2$.

17. $64a^2b^2 - 36$.

18. $49v^4 - 100w^4$.

19. $121c^2 - 144d^2$.

20. $3abc^2 - 48abx^2$.

21. $52c^2x^2 - 117c^2z^4$.

22. $a^2 - b^2 - c^2 + 2bc$.

23. $2ab - b^2 + 1 - a^2$.

24. $12bc - 9c^2 - 4b^2 + a^2$.

25. $a^2x^2 - 1 + 2abxy + b^2y^2$.

26. $c^2 + a^2 - d^2 - b^2 + 2bd - 2ac$.

27. $a^2 - d^2 - c^2 + b^2 - 2cd - 2ab$.

28. $a^2 + b^2 - m^2 - n^2 - 2ab - 2mn$.

29. $169a^4b^4 - 196s^4$.

30. $225d^8 - 256f^8$.

31. $81x^{10} - 289u^{10}$.

32. $121g^{12} - 324h^{12}$.

33. $49k^{14} - 361m^{14}$.

34. $400n^{16} - 484p^{16}$.

35. $441q^{18} - 625r^{18}$.

36. $625s^{20} - 841t^{20}$.

37. $(a - b)^2 - 1$.

38. $1 - (g - 3h)^2$.

39. $(2ab - a^2 - b^2) + 1$.

40. $a^4 + 2a^2b^2 + b^4 - 9a^2b^2$.

41. $(3x - 2y)^2 - (4a + 5b)^2$.

42. $(3c + 4d)^2 - (5e - 6f)^2$.

43. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$.

44. $(2x - 3y)^2 - (3x^2 - 2y^2 - z^3)^2$.

45. $2by + a^2 - y^2 - 2ax - b^2 + x^2$.

46. $a^2x^2 + 2abxy + b^2y^2 - 1 - 2cd - c^2d^2$.

47. $(a^2 - 2ab + b^2) - (m^2 + n^2 + 2mn)$.

48. $64a^2b^2 - (a^2 + 16b^2 - c^2)^2$. Check.

49. $(x^2 + 9y^2 - z^2)^2 - 36x^2y^2$. Check.

50. $(2x^2 + 3x - 7)^2 - (x^2 - 5x + 20)^2$.

51. $4x^4 + 12x^2y^2 + 9y^4 - 49x^2y^2$.

52. $4(xy + ab)^2 - (x^2 - a^2 - b^2 + y^2)^2$.

53. $9x^2 - 12xy + 4y^2 - 16a^2 - 8ab - b^2$.

205. Cor. 2. A polynomial of the form $x^{4n} + ax^{2n}y^{2n} + y^{4n}$ can be resolved into two trinomial factors by first reducing it to the difference of two squares by Prop. 3, and then factoring by Prop. 4.

Model Solutions

1. Factor $4x^4+3x^2a^2+9a^4$.

$4x^4+3x^2a^2+9a^4=(4x^4+12x^2a^2+9a^4)-9x^2a^2$ by adding and also subtracting $9x^2a^2$. For the trinomial to be a perfect square, its middle term must be twice the product of the square roots of the first and third terms, according to Prop. 3. $3x^2a^2=12x^2a^2-9x^2a^2$. What is twice the product of the square roots of the first and third terms? What must be added to $3x^2a^2$? What else must be done? Why?

$$\begin{aligned}(4x^4+12x^2a^2+9a^4)-9x^2a^2&=(2x^2+3a^2)^2-(3xa)^2\\&=(2x^2+3a^2+3xa)(2x^2+3a^2-3xa).\end{aligned}$$

2. Factor $x^{4n}+y^{4n}$. $\quad(\equiv x^{4n}+0\,x^{2n}y^{2n}+y^{4n}.)$

$$\begin{aligned}x^{4n}+y^{4n}&=x^{4n}+2x^{2n}y^{2n}+y^{4n}-2x^{2n}y^{2n}\\&=(x^{2n}+y^{2n})^2-2x^{2n}y^{2n}\\&=(x^{2n}+y^{2n}+x^ny^n\sqrt{2})(x^{2n}+y^{2n}-x^ny^n\sqrt{2}).\end{aligned}$$

206. Cor. 3. A trinomial of the form $x^{4n}+x^{2n}y^{2n}+y^{4n}$, in which all the terms are positive squares and one term is the product of the square roots of the other two, can be resolved into two trinomial factors, which will be:

1. The sum of the square roots of the three terms.

2. The sum of the square roots of the first and third terms minus the square root of the middle term.

Dem. This is but a special caſe of Cor. 2, in which a, the coefficient of x^2y^2, must be a positive square equal to the product of the square roots of the other two terms.

Note. Since this form is of frequent occurrence in algebra, the student will find this corollary very convenient in writing the factors immediately. Thus, $x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2)$. What similarity do you notice? What difference? Why does not this corollary come directly under Prop. 3?

Model Solution

Factor $x^8 + x^4y^4 + y^8$.

$x^8 + x^4y^4 + y^8 = (x^4 + x^2y^2 + y^4)(x^4 - x^2y^2 + y^4)$, by Cor. 3.

$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2)$, by Cor. 3.

$$x^4 - x^2y^2 + y^4 = (x^4 + 2x^2y^2 + y^4) - 3x^2y^2$$
$$= (x^2 + y^2 + xy\sqrt{3})(x^2 + y^2 - xy\sqrt{3}).$$

EXAMPLES

Factor:

1. $x^4 + x^2 + 1$.
2. $x^4 + 16x^2 + 256$.
3. $x^4 + 9x^2 + 81$.
4. $x^{16} + x^8y^8 + y^{16}$.
5. $1 + Q^2 + Q^4$.
6. $144a^4 + 36a^2b^2 + 9b^4$.
7. $16x^4 + 4x^2a^2 + a^4$.
8. $x^4 + a^4$. Check.
9. $x^4 + 1$. Check.
10. $9x^4 - 4x^2z^2 + 4z^4$.
11. $25a^4 - 36a^2c^2 + 256c^4$.
12. $x^8 + y^8$.
13. $a^4 + 4x^4$.
14. $4c^4 - 36c^2d^2 + 25d^4$.
15. $m^{4n} + m^{2n}s^{2n} + s^{4n}$.
16. $49s^4 + 66s^2t^2 + 25t^4$.
17. $16x^4 + 36x^2w^2 + 81w^4$.
18. $v^4 - 17v^2w^2 + 16w^4$.
19. $144x^4 - 169x^2y^2 + 49y^4$.
20. $x^{12} + y^{12}$. Check.
21. $36h^4 - 28h^2k^2 + 4k^4$.
22. $x^4 - 7x^2 + 1$. Check.

207. Prop. 5. A trinomial of the form $acx^2 + (ad + bc)x + bd$ can be resolved into two binomial factors, $(ax + b)(cx + d)$, by factoring the first and third terms into two factors each, so that the sum of the products of the extremes and the means shall equal the second term of the trinomial.

Dem. See Special Theorem 5.

Model Solutions

1. Factor $10x^2 - 11x - 6$.

$$\begin{array}{rcr} 2x & - & 3 \\ 5x & + & 2 \\ \hline & & -15x \\ & & +\ 4x \\ \hline & & -11x \end{array}$$

$$\therefore\ 10x^2 - 11x - 6 = (5x + 2)(2x - 3).$$

Explanation. The factors of the first term are $5x$ and $2x$. The factors of the third term, arranged so that the sum of the diagonal products shall be $-11x$, are $+2$ and -3. 2 and 3 must have opposite signs because 6 is negative. 3 must be negative because the larger diagonal product must be minus to produce a negative middle term. $-15x + 4x = -11x$. $\therefore$ the factors are $(5x + 2)(2x - 3)$.

If the factors are written side by side, as $(5x + 2)(2x - 3)$, the middle term of the product is found by adding the products of the extremes and the means. $4x - 15x = -11x$.

Note. This proposition requires one to factor by inspection. The general case is treated in Prop. 9. It takes some practice and considerable good judgment to be able to tell readily the right set of factors for the separate terms, their proper arrangement, and their correct signs. It may be helpful to notice that when the sign of the last term of the arranged trinomial is positive, its factors are positive if the middle term is positive, and negative if the middle term is negative; also, that when the last term is negative its factors are unlike in signs, the sign of the middle term being given to the factor of the third term forming the greater diagonal product.

2. Factor $6(x^2 - x)^2 - 5(x^2 - x) - 21$.

$$\begin{array}{r} 2(x^2 + x) + 3 \\ 3(x^2 - x) - 7 \\ \hline +\ 9 \\ -\ 14 \\ \hline -\ 5 \end{array}$$

EXAMPLES

1. $3a^2 - 2a - 1.$
2. $5a^2 - 8ax + 3x^2.$
3. $8a^2 + 22ab + 15b^2.$
4. $8x^2 - 6xy - 5y^2.$
5. $12x^2 + 55x + 63.$
6. $7x^2 - 12x + 5.$
7. $20x^4 + x^2 - 1.$
8. $14x^2 + 5x - 1.$
9. $2x^2 + 3xy + y^2.$
10. $12z^2 + 5z - 3.$
11. $6a^2 - a - 2.$
12. $6a^2 + 11ax + 3x^2.$
13. $21x^2 - 26x + 8.$
14. $21y^2 - 17y + 2.$
15. $6a^2 + 7ax - 3x^2.$
16. $ax^2 - dx + cd - acx.$
17. $4x^2 - 28xy + 48y^2.$
18. $12a^2b + 3by^2 - 15aby.$
19. $6(x+y)^2 - 31(x+y) + 35.$
20. $21(a+b+c)^2 - 17(a+b+c) - 30.$
21. $10(a+b)^2 - (a+b)(c-d) - 3(c-d)^2.$
22. $(4a^2 - 9b^2)x^2 + (4a^2b^2 + 4ac^2d)x + (ab^2 + c^2d)^2.$

208. Cor. A trinomial of the form $x^2 + (a+b)x + ab$ can be resolved into two binomial factors $(x+a)$ and $(x+b)$ by factoring the first term, and then factoring the last term into two factors such that their sum shall be equal to the coefficient of x in the second term.

Dem. See Special Theorem 4. This is but a special case of Prop. 5, in which the coefficient of x^2 is unity.

Model Solution

Factor $x^2 + 2x - 48.$

$$\begin{array}{ccc} x & + & 8 \\ x & - & 6 \\ \hline \multicolumn{3}{c}{8x - 6x = 2x} \end{array}$$

The factors of -48 whose sum is $+2$ are $+8$ and -6. Hence the factors are $(x+8)$ and $(x-6)$.

MENTAL EXERCISES

Factor:

1. $x^2 - 2x - 35$.
2. $x^2 - 6x - 91$.
3. $x^2 - 7x + 6$.
4. $x^2 - 3x - 28$.
5. $x^2 - 11x + 28$.
6. $x^2 + 2x - 35$.
7. $x^2 - 2x - 8$.
8. $x^2 + 2x - 8$.
9. $72 - 6y - y^2$.
10. $5 + 6s + s^2$.
11. $x^2 + 12x - 28$.
12. $P^2 - P - 72$.
13. $a^2 + 9ab + 8b^2$.
14. $y^6 - 4y^3 + 3$.
15. $x^2 - 8x - 65$.
16. $x^4 - 11x^2 - 12$.
17. $x^6 - x^3 - 12$.
18. $x^2 + 29x - 390$.
19. $29 - 28x - x^2$.
20. $m^2 - 2m - 323$.
21. $x^4 + 20x^2 + 19$.
22. $a^2 - 24a + 143$.
23. $a^2 + \frac{1}{6}a - \frac{1}{6}$.
24. $5R^2 - 20R + 20$.
25. $c^4 - c^2 - 132$.
26. $z^2 + (a - b)z - ab$.
27. $w^2 - (a - b)w - ab$.
28. $Q^2 + 2Q - 63$.
29. $m^2 + 29m + 100$.
30. $n^2 + n - 56$.
31. $k^2 + 25kx - 150x^2$.
32. $x^2 + 17x - 390$.
33. $x^6 - x^3 - 2$.
34. $x^2 + 57x + 56$.
35. $x^8 - 2x^4 - 24$.
36. $25x^3 - 30x^2y + 9xy^2$.
37. $c^2 - 2c - 399$. Check.
38. $2xy^6 - 10x^2y^3 - 28x^3$.
39. $a^2x^2 - 6axyz - 91y^2z^2$.
40. $a^2x^2 + 3ax - 154$. Check.
41. $x^2 + 7(a + b)x + 10(a + b)^2$.
42. $7x^2 - 21(c-d)x - 280(c-d)^2$.

209. Prop. 6. The sum of two numbers is a divisor of the sum of their same odd powers, and of the difference of their same even powers; the difference of two numbers is a divisor of the difference of the same powers of those numbers.

Dem. See Special Theorem 11 and demonstration.

Review the exercises under Art. **185**.

MENTAL EXERCISES

Factor:

1. $x^3 - a^3$.
2. $x^3 + a^3$.
3. $x^3 - 125\,y^3$.
4. $4\,x^3 - 32\,y^3$.
5. $a^3 + \frac{1}{27}\,b^3$.
6. $x^6 + y^6$.
7. $x^6 - y^6$.
8. $32\,c^5 + y^5$.
9. $h^{24} + J^{24}$.
10. $c^{12} - d^{12}$.
11. $x^7 + y^7$.
12. $a^9 - b^9$.
13. $x^4 - y^4$.
14. $64\,a^6 + 2\,ax^5$.
15. $8\,z^3 - 27\,w^3$.
16. $343\,t^6 - u^3$.
17. $x^6 + 64\,b^6$.
18. $32\,r^5 + 243\,s^5$.
19. $64\,m^3 - 125\,n^3$.
20. $1728\,c^3 + 64\,d^3$.
21. $8\,E^3 + 27\,F^3$.
22. $(x+y)^3 - 8\,z^3$.
23. $(2\,x - y)^3 - (x + 2\,y)^3$.
24. $(x + 2\,y)^3 + (2\,x - y)^3$.
25. $(x+y)^3 - z^3$. Check.
26. $(a-b)^3 + 27$. Check.
27. $(a-b)^3 - (c-d)^3$.
28. $x^{10} + y^{10}$.
29. $8\,x^3 + 1$.
30. $88(a+b)^3 + 297$.
31. $27\,(x^2 - y^2)^3 - 8\,z^3$.
32. $(x+y)^6 - (x-y)^6$.
33. $(2\,x - 3\,y)^3 + (3\,x)^3$.

210. Prop. 7. A polynomial of the form $Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + L$ can sometimes be factored by dividing it by some factor of A times the first power of the letter of arrangement, plus or minus some factor of the last term.

Dem. This is simply an application of the process of synthetic division. See Art. **177**.

MODEL SOLUTION

Factor $15\,x^3 + 41\,x^2 + 5\,x - 21$.

$$
\begin{array}{rrrr|l}
15 & +41 & +5 & -21 & \underline{+\frac{7}{5}} \\
 & +35 & +14 & -21 & \\
\hline
15 & +6 & -9 & \multicolumn{2}{|l}{\underline{-\frac{3}{5}}} \\
 & -9 & -9 & & \\
\hline
15 & +15 & \multicolumn{3}{|l}{\underline{+1}} \\
 & +15 & & & \\
\hline
15 & & & &
\end{array}
$$

$$\begin{aligned} \therefore\ 15\,x^3 + 41\,x^2 + 5\,x - 21 &= 15(x + \tfrac{7}{3}) \cdot (x - \tfrac{3}{5}) \cdot (x + 1) \\ &= 3(x + \tfrac{7}{3}) \cdot 5(x - \tfrac{3}{5}) \cdot (x + 1) \\ &= (3\,x + 7) \cdot (5\,x - 3) \cdot (x + 1). \end{aligned}$$

Explanation. In this polynomial $A = 15$ and $L = -21$. The factors of 15 are 3 and 5; of -21, $+7$ and -3, or -7 and $+3$. By trial, $3\,x + 7$ proves to be the exact divisor. $5\,x - 3$ is the divisor of $15\,x^2 + 6\,x - 9$, and the quotient, or last factor, is $x + 1$. The factor $x + 1$ would naturally be found first, but better to illustrate the principles involved in the proposition, the other factors are found first.

Note. This proposition is of special importance in reducing polynomials of higher degrees to forms of lower degree so that other propositions may be brought to bear upon them.

EXAMPLES

Factor:

1. $x^3 - 8\,x^2 + 5\,x + 14.$
2. $x^3 - 9\,x^2 + 26\,x - 24.$
3. $x^3 - 6\,x^2 + 11\,x - 6.$
4. $x^4 + x^3 - 4\,x^2 + x + 1.$
5. $x^4 - 8\,x^3 + 21\,x^2 - 20\,x + 4.$
6. $3\,x^3 + 8\,x^2 + 5\,x + 2.$
7. $3\,x^3 + 9\,x^2 + 9\,x + 6.$
8. $2\,x^3 - 12\,x^2 + 21\,x - 10.$
9. $x^3 - 2\,ax^2 + 4\,xa^2 - 8\,a^3.$
10. $x^3 + 2\,ax^2 + 4\,a^2x + 8\,a^3.$
11. $x^6 - 1$ and $x^5 + 1.$
12. $x^3 - 1$ and $x^5 - 1.$
13. $x^3 + 1$ and $x^4 - 1.$
14. $x^3 - 9\,x^2 + 23\,x - 15.$
15. $3\,x^5 - 10\,x^3 + 15\,x + 8.$
16. $x^4 - 4\,x^3 + 6\,x^2 - 4\,x + 1.$
17. $2\,x^4 + 17\,x^3 + 0\,x^2 - 68\,x - 32.$
18. $x^4 - 16\,x^3 + 86\,x^2 - 176\,x + 105.$
19. $x^5 - 2\,x^4 - 6\,x^3 + 4\,x^2 + 13\,x + 6.$
20. $x^5 + 5\,x^4y + 10\,x^3y^2 + 10\,x^2y^3 + 5\,xy^4 + y^5.$
21. $x^3 + (a + b + c)\,x^2 + (ab + ac + bc)\,x + abc.$
22. $x^3 + (a - b - c)\,x^2 - (ab + ac - bc)\,x + abc.$

211. Prop. 8. A polynomial of the form $Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + L$ can sometimes be resolved into two polynomial factors by:

1. Factoring the first and the last terms, and the letters of the intermediate terms having even exponents.

2. Determining the coefficients of the new intermediate terms by the use of the terms having odd exponents.

3. Making a final test by the use of the terms having even exponents.

Dem. The principle involved is the same as found in Prop. 5. It involves multiplication by diagonal inspection.

Model Solutions

1. Factor $2x^4 + 5x^3 - 10x^2 + 14x - 15$.

$$\begin{array}{ccc} 2x^2 & -\ x & +3 \\ x^2 & +3x & -5 \end{array}$$

Explanation. 1. Factor first and last terms as in Prop. 5, and the letters of the intermediate term, $-10x^2$ (not the coefficient of it).

2. Determine the coefficients of the *new* intermediate terms, x and x, by the use of the terms having odd exponents, *i.e.*, $5x^3$ and $+14x$. This gives rise to two questions: What coefficients must be given to x and x that the sum of the first diagonal products shall be $+5x^3$, and of the second $+14x$? By trial they are found to be -1 and $+3$.

3. The final test is that the sum of the extreme diagonal products together with the product of the middle terms must equal $-10x^2$. The extreme diagonal products are $-10x^2$ and $+3x^2$; the product of the middle terms is $-3x^2$; the sum of the products is $-10x^2$. Thus all the terms of the polynomial have been accounted for, and may be checked off. Hence the factors are $(2x^2 - x + 3)(x^2 + 3x - 5)$.

2. Factor $x^4 + 4x^3 + 0x^2 - 8x + 4$.

$$\text{1st factor} = x^2 + 2x - 2.$$

$$\text{2d factor} = x^2 + 2x - 2.$$

3. Factor $9a^4 - 4a^2b^2 + 4b^4$.

$$9a^4 - 4a^2b^2 + 4b^4 \equiv 9a^4 + 0a^3b - 4a^2b^2 + 0ab^3 + 4b^4.$$

$$\text{1st factor} = 3a^2 + 4ab + 2b^2.$$

$$\text{2d factor} = 3a^2 - 4ab + 2b^2.$$

Explanation. In this example the sum of the extreme diagonal products is $+12a^2b^2$; the problem requires $-4a^2b^2$. The question is, therefore, what must be added to $+12a^2b^2$ to give $-4a^2b^2$. This is evidently $-16a^2b^2$. Since the coefficients of the odd powers are 0, $-16a^2b^2$ must be divided into two numerically equal factors, *i.e.*, $+4ab$ and $-4ab$.

4. Factor $6x^5 - 9x^4 + 19x^3 - 12x^2 + 19x - 15$.

$$\text{1st factor} = 3x^3 + 0x^2 + 2x - 3.$$

$$\text{2d factor} = 0x^3 + 2x^2 - 3x + 5.$$

5. Factor $ax + ax^2 + ax^3 - 1 - x - 2x^2 - x^3 - x^4$.

$$ax + ax^2 + ax^3 - 1 - x - 2x^2 - x^3 - x^4 \equiv -1 + (a-1)x + (a-2)x^2 + (a-1)x^3 - x^4.$$

$$\text{1st factor} = +1 + x + x^2.$$

$$\text{2d factor} = -1 + ax - x^2.$$

EXAMPLES

Factor:

1. $3x^4 - 14x^3 - 9x + 2$.
2. $2x^4 - 9x^3 - 14x + 3$.
3. $x^4 - ax^3 - a^3x + a^4$.
4. $x^4 - 5x^3 + 20x - 16$.
5. $6x^4 - x^3 - 3x^2 - 4x - 4$.
6. $x^4 + 2x^3 - 7x^2 + 8x - 5$.
7. $x^4 - 2x^3 - 3x^2 + 8x - 4$.
8. $x^4 - 2x^3 + 3x^2 - 2x - 3$.
9. $3x^4 + 2x^3 - 3x^2 + 2x - 1$.
10. $6x^4 - 2x^3 - 3x^2 + 4x - 1$.
11. $8x^4 + 2x^3 - 4x^2 + 3x + 1$.
12. $15x^4 - 2x^3 + 10x^2 - x + 2$.
13. $4x^4 - 2x^3 + 10x^2 + x + 15$.
14. $x^4 - 8x^3 + 21x^2 - 20x + 4$.
15. $8x^4 - 2x^3 + 23x^2 + 3x + 10$.
16. $x^4 - x^3 - 39x^2 + 24x + 180$.
17. $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6$.
18. $x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2$.

212. Prop. 9. Any trinomial of the form $Ax^2 + Bx + C$ can be reduced to the difference of two squares (by Prop. 3) and resolved into factors, which shall be rational and integral with respect to the unknown quantity (by Prop. 4).

The demonstration will be better understood after a few illustrations, and may be omitted until Quadratic Equations are reached.

Model Solutions

1. Factor $x^2 + 10x + 24$.

In order that a trinomial of this form shall be a perfect square, its third term must be the square of one half the coefficient of the second term. $(\frac{1}{2} \text{ of } 10)^2 = 25$. Hence

$$x^2 + 10x + 24 \equiv x^2 + 10x + 25 - 25 + 24, \text{ by adding and subtracting } 25$$
$$= (x^2 + 10x + 25) - (25 - 24), \text{ by grouping}$$
$$= (x+5)^2 - 1^2, \text{ by Prop. 3}$$
$$= ((x+5)+1)((x+5)-1), \text{ by Prop. 4}$$
$$= (x+6)(x+4), \text{ the required factors.}$$

2. Factor $2x^2 - 5x + 2$.

$$2x^2 - 5x + 2 \equiv 2(x^2 - \tfrac{5}{2}x + 1), \text{ by multiplying and dividing by } 2$$
$$= 2(x^2 - \tfrac{5}{2}x + \tfrac{25}{16} - \tfrac{25}{16} + 1), \text{ by adding and subtracting } (\tfrac{1}{2} \text{ of } \tfrac{5}{2})^2, \text{ or } \tfrac{25}{16}$$
$$= 2\{(x - \tfrac{5}{4})^2 - \tfrac{9}{16}\}, \text{ by Prop. 3}$$
$$= 2\{x - \tfrac{5}{4} + \tfrac{3}{4}\}\{x - \tfrac{5}{4} - \tfrac{3}{4}\}, \text{ by Prop. 4}$$
$$= 2\{x - \tfrac{1}{2}\}\{x - 2\}, \text{ by combining terms}$$
$$= (2x - 1)(x - 2), \text{ the required factors.}$$

3. Factor $x^2 + 4x + 6$.

$$x^2 + 4x + 6 \equiv x^2 + 4x + 4 - 4 + 6, \text{ by adding and subtracting } (\tfrac{1}{2} \text{ of } 4)^2, \text{ or } 4$$
$$= (x+2)^2 - (-2), \text{ by Prop. 3}$$
$$= (x + 2 + \sqrt{-2})(x + 2 - \sqrt{-2}), \text{ by Prop. 4.}$$

213. Upon the principle that the middle term of a trinomial square is twice the product of the square roots of the other two, it is often possible to complete the square more advantageously than by the ordinary method as shown by the three preceding solutions.

4. Factor $4x^2 + 16x - 33$.

$4x^2 + 16x - 33 \equiv 4x^2 + 16x + 16 - 16 - 33$, by dividing $16x$ by twice $\sqrt{4x^2}$, squaring the result, and adding and subtracting,

$= (4x^2 + 16x + 16) - (16 + 33)$, by grouping

$= (2x + 4)^2 - 49$, by Prop. 3

$= (2x + 4 + 7)(2x + 4 - 7)$, by Prop. 4

$= (2x + 11)(2x - 3)$, the desired factors.

5. Factor $3x^2 + 2x - 5$.

$$\begin{aligned}
3x^2 + 2x - 5 &\equiv \tfrac{1}{3}(9x^2 + 6x - 15) \\
&= \tfrac{1}{3}(9x^2 + 6x + 1^2 - 1^2 - 15) \\
&= \tfrac{1}{3}\{(9x^2 + 6x + 1^2) - (1 + 15)\} \\
&= \tfrac{1}{3}\{(3x + 1)^2 - 16\} \\
&= \tfrac{1}{3}\{3x + 1 + 4\}\{3x + 1 - 4\} \\
&= \tfrac{1}{3}(3x + 5)(3x - 3) \\
&= (3x + 5)(x - 1).
\end{aligned}$$

6. Factor $3x^2 - 5x - 4$.

$$\begin{aligned}
3x^2 - 5x - 4 &\equiv \tfrac{1}{2}(6x^2 - 10x - 8) \\
&= \tfrac{1}{12}(36x^2 - 60x + 25 - 25 - 48) \\
&= \tfrac{1}{12}\{(6x - 5)^2 - 73\} \\
&= \tfrac{1}{12}\{6x - 5 + \sqrt{73}\}\{6x - 5 - \sqrt{73}\}.
\end{aligned}$$

Dem. of Prop. 9.

1. $Ax^2 + Bx + C \equiv A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right)$. Why?

$= A\left(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2} - \frac{B^2}{4A^2} + \frac{C}{A}\right)$. Why?

$= A\left\{\left(x + \frac{B}{2A}\right)^2 - \left(\frac{B^2 - 4AC}{4A^2}\right)\right\}$, by Prop. 3

$= A\left\{x + \frac{B}{2A} + \sqrt{\frac{B^2 - 4AC}{4A^2}}\right\}\left\{x + \frac{B}{2A} - \sqrt{\frac{B^2 - 4AC}{4A^2}}\right\}$, by Prop. 4

$= A\left\{x + \frac{B + \sqrt{B^2 - 4AC}}{2A}\right\}\left\{x + \frac{B - \sqrt{B^2 - 4AC}}{2A}\right\}$.

Since these factors are rational and integral with respect to the unknown quantity, the truth of the proposition appears.

2. $Ax^2 + Bx + C \equiv \frac{1}{4A}\{4A^2x^2 + 4ABx + 4AC\}$. Why?

$= \frac{1}{4A}\{4A^2x^2 + 4ABx + B^2 - B^2 + 4AC\}$. Why?

$= \frac{1}{4A}\{(2Ax + B)^2 - (B^2 - 4AC)\}$, by Prop. 3

$= \frac{1}{4A}\{2Ax + B + \sqrt{B^2 - 4AC}\}\{2Ax + B - \sqrt{B^2 - 4AC}\}$, by Prop. 4.

EXAMPLES

Factor:

1. $x^2 + 5x + 6$.
2. $x^2 - 7x + 6$.
3. $x^2 - 2x - 8$.
4. $x^2 + 12x - 28$.
5. $x^2 + x - 56$.
6. $x^2 - 11x + 28$.
7. $3x^2 - 2x - 1$.
8. $7x^2 - 12x + 5$.
9. $14x^2 + 5x - 1$.
10. $12x^2 + 5x - 3$.
11. $6y^2 + 7y - 3$.
12. $8x^2 - 6xy - 5y^2$.
13. $ax^2 - bx + c$.
14. $x^2 + 3x + 4$.
15. $2x^2 - 3x - 4$.
16. $5x^2 + 6x + 7$.
17. $7x^2 - 8x - 9$.
18. $6(x+y)^2 - 31(x+y) + 35$.

214. Cor. Instead of working any example under Prop. 9, the general results obtained in either one of the demonstrations may be used as a **formula**, and the values of A, B, and C in any particular example substituted therein. The results, when simplified, will be the required factors.

Model Solution

Factor $3x^2 + 2x - 5$ by substituting in the second formula. In this example $A = 3$, $B = +2$, $C = -5$.

Substituting these values,

$$\frac{1}{4A}\{2Ax + B + \sqrt{B^2 - 4AC}\}\{2Ax + B - \sqrt{B^2 - 4AC}\} \text{ becomes}$$

$$\begin{aligned} &\tfrac{1}{12}\{6x + 2 + \sqrt{4 + 60}\}\{6x + 2 - \sqrt{4 + 60}\} \\ &= \tfrac{1}{12}(6x + 2 + 8)(6x + 2 - 8) \\ &= \tfrac{1}{12}(6x + 10)(6x - 6) \\ &= \tfrac{1}{2}(6x + 10) \cdot \tfrac{1}{6}(6x - 6) \\ &= (3x + 5)(x - 1). \end{aligned}$$

EXAMPLES

Factor the following expressions by substituting in either formula:

1. $x^2 - 8x + 15$.
2. $x^2 - 2x - 35$.
3. $10x^2 - 11x - 6$.
4. $21x^2 - 17x + 2$.
5. $12y^2 + 5y - 3$.
6. $2x^2 + 3x + 4$.
7. $5x^2 - 6x + 7$.
8. $8x^2 + 9x - 10$.
9. $x^4 - 26x^2 + 25$.
10. $11x^2 - 12x - 13$.
11. $3(x + y)^2 - 2(x + y) - 1$.
12. $2x^2 - 3x - 4$.
13. $4x^2 + 2x - 3$.
14. $3x^2 - 4x + 2$.
15. $5x^2 + 6x + 7$.
16. $6x^2 - 5x + 7$.
17. $7x^2 + 5x - 6$.
18. $1 - 2x + 3x^2$.
19. $2 + 3x^2 + 4x$.
20. $ax^2 - bx - c$.
21. $bx - ax^2 + c$.
22. $-c - ax^2 - bx$.

MISCELLANEOUS MENTAL EXERCISES

Factor and state propositions used:

1. $x^2 - a^2$.
2. $x^3 + a^3$.
3. $x^3 - 1$.
4. $x^5 + a^5$.
5. $x^6 + a^6$.
6. $x^6 - a^6$.
7. $x^2 + 4xy + 4y^2$.
8. $16x^2 - 8xa + a^2$.
9. $x^4 - 26x^2 + 25$.
10. $2x^2 - 7x + 6$.
11. $(a-b)^2 - (c-d-e)^2$.
12. $2xy + y^2 - z^2 + x^2$.
13. $x^2 - 2xz + z^2 - y^2 - 2ys - s^2$.
14. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
15. $(ax - by)^2 - 1$.
16. $6x^2 + x - 2$.
17. $x^4 - a^4$. Check.
18. $3a^2bc^3 - 24a^2bd^3$.
19. $x^2 - y^2 + x - y$.
20. $x^{10} + y^{10}$. Check.
21. $10x^2 - 11x - 6$.
22. $x^2 - 5xa + 4a^2$.
23. $x^2 + 87x + 86$.
24. $x^4 + x^2y^2 + y^4$.
25. $8x^3 - 1$. Check.
26. $\Sigma a^2 + \Sigma 2ab$.
27. $a^3 + b^3 + 3a^2b + 3ab^2$.
28. $\Sigma a^3 + \Sigma 3a^2b + \Sigma 6abc$.

MISCELLANEOUS EXAMPLES

Resolve into prime factors:

1. $x^2 + 8x - 33$.
2. $x^2 - 12x + 27$.
3. $3x^2 - 7x - 40$.
4. $x^{10} + 25x^5 - 26$.
5. $x^3 - 6x^2 + 11x - 6$.
6. $x^4 + x^3 - 4x^2 + x + 1$.
7. $x^4 + x^2 - 2$. Check.
8. $8x^2 - 18xy - 5y^2$.
9. $15a^3b^3y^2 + 25a^2b^3y^3$.
10. $x^3 + y^3 + 3x^2y + 3xy^2$.
11. $x^4 - 16$.
12. $x^5 + 32$.
13. $11x^2 + 34x + 3$.
14. $x^8 + x^4a^4 + a^8$.
15. $25s^4 - 36s^2t^2 + 4t^4$.
16. $x^3 - 9x^2 + 26x - 24$.
17. $Ax^2 - Bx + C$.
18. $9x^4 - 30x^2a + 25a^2$.
19. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
20. $6x^5 + 15x^4y - 4x^3c^2 - 10x^2yc^2$.
21. $a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3a^2c + 3ac^2 + 3b^2c + 3bc^2 + 6abc$.

22. $a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$.

23. $s^4 + t^4 - a^4 - b^4 + 2s^2t^2 - 2a^2b^2$.

24. $ab^2 - abx - axy + aby + b^2x + bxy - bx^2 - x^2y$.

25. $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$.

26. $x^4 - 2x^3 + 3x^2 - 2x - 3$.

27. $x^3 - 33x^2 + 92x - 60$.

28. $35a^8b^2c^4x^2y^3$. Check.

29. $(a - x^2)^2 - (b - y^2)^2$.

30. $x^3 + y^3 + x + y$.

31. $c^5d^3 - c^2 - c^3d^3x^2 + x^2$.

32. $5z^4 - 15z^3 - 90z^2$.

33. $x^2 - 9y^2 + x + 3y$.

34. $x^4 + 4x^3 + 8x^2 + 8x + 4$.

35. $a^3 - 2a^2b + 4ab^2 - 8b^3$.

36. $48x^4 + 16x^2 - 15$.

37. $4x^3 - 27x + 5$. Check.

38. $(y^2 + y)^2 - 2(y^2 + y) - 8$.

39. $16x^4 - y^2 - 6x^2y$.

40. $x^4 - 2x^2y^2 + y^4 - x^2 + y^2$.

41. $x^4 + y^4 - 12x^3 + 12y^3$.

42. $4x^4 + 16x^3 - 57x^2 + 16x + 4$.

43. $a^2x^2 - 2a^3x + a^4 - 1$.

44. $4a^{4m} - 4a^{2m}b^n + b^{2n}$.

45. $x^{2n} - x^n + \frac{1}{4}$. Check.

46. $x^3 + 5x^2 - 9x - 45$.

47. $x^2 - 2ax - 2bx + 4ab$.

48. $x^2 - 2xy + y^2 - z^2$.

49. $(3a + 2b)^2 - (2b - 3a)^2$.

50. $a^2(b-c) + b^2(c-a) + c^2(a-b)$.

51. $4x^4 - 8x^3y^2 + 4xy^6 + y^8$.

52. $aby^2 - axy + bxy - x^2$.

53. $x^2 - a^2 - b^2 + y^2 - 2xy - 2ab$.

54. $ax^5 - by^5 - ay^5 + by^5$.

55. $x^4 - 11x^2y^2 + y^4$. Check.

56. $x^3 - 9x^2 + 26x - 24$.

57. $a^2 + 2ab + b^2 + 3a + 3b - 10$.

58. $x^4 - 47x^2y^2 + y^4$.

59. $x^3 - 8x^2 + 5x + 14$.

60. $2xy - x^2 - y^2 + 1$.

61. $9a^4 + 21a^2b^2 + 25b^4$.

62. $2x^2 + 5xy + 2y^2 - 8x - 7y + 6$.

63. $6x^2 - 5xy - 6y^2 + 4x + 7y - 2$.

64. $a^3 - 2a^2b + a^2 - 4a + 8b - 4$.

65. $x^3 - 18x^2 + 107x - 210$.

Section II

EQUATIONS SOLVED BY FACTORING

215. Zero Factors. Since zero is defined as a difference in the form $a - a$, a zero factor in any product may be replaced by $a - a$.

216. Prop. 1. The product of two or more factors is zero when, and only when, one or more of the factors is zero.

Dem. Let F represent any factor of finite value. Then the product $0 \times F = (a-a)F = aF - aF = 0$.

Hence the product is zero when one of two factors is zero, and it is obvious that it is not zero unless one factor is zero.

Let the student prove the proposition when two of three factors are equal to zero.

217. Prop. 2. An equation which can be reduced to the form $(x-a)(x-b)(x-c)\cdots = 0$ may be solved by putting each factor equal to zero.

Dem. By Prop. 1, the product $(x-a)(x-b)(x-c)\cdots$ is zero when $x-a=0$, or $x-b=0$, or $x-c=0$, etc. Hence the equation is satisfied if $x-a=0$, or $x-b=0$, or $x-c=0$, etc., and in no other case. Solving by transposition, $x=a$, $x=b$, $x=c$, $\cdots$.

218. Cor. *Formation of Equations.* An equation in x whose roots are a, b, c, etc., may be formed by subtracting each of the roots from x and putting the product of these results equal to zero.

For, since $x=a$, $x=b$, $x=c$, etc., $x-a$, $x-b$, $x-c$, etc., each is equal to zero. Hence their product is equal to zero. But this is the equation whose roots are a, b, c, etc., and hence the one desired.

Model Solutions

1. Solve $3x^2 + 2x - 5 = 0$.

1. $(x-1)(3x+5) = 0$, by factoring.
2. $x - 1 = 0$, by Prop. 1.
3. $3x + 5 = 0$. Why?
4. $\therefore x = 1$ and $-\frac{5}{3}$.

2. Form an equation, the roots of which are 1, -2, and 3.

Let x represent the unknown quantity in the equation. Since 1, -2, and 3 represent the values of the unknown quantity which satisfy the equation, $x = 1$, $x = -2$, $x = 3$. By transposition, $x - 1 = 0$, $x + 2 = 0$, $x - 3 = 0$.

By Prop. 1, $(x-1)(x+2)(x-3) = 0.$

Hence, $x^3 - 2x^2 - 5x + 6 = 0$ is the equation.

Do the three x's in $(x-1)(x+2)(x-3) = 0$ have the same or different values? What is the value of x in $(x-1)$? in $(x+2)$? in $(x-3)$?

EXAMPLES

Solve by method of factoring:

1. $x^2 - 4x + 4 = 0.$
2. $6x^2 - 7x - 20 = 0.$
3. $3x^2 - 10x + 3 = 0.$
4. $2x^2 + 3x + 1 = 0.$
5. $x^2 - x - 2 = 0.$
6. $x^2 + 11x + 30 = 0.$
7. $x^3 - 2x^2 = 15x + 36.$
8. $3x^2 - 4x - 15 = 0.$
9. $x^2 - 2xa + a^2 = 0.$
10. $2x^2 - x - 6 = 0.$
11. $x^3 - x^2 - 7x + 3 = 0.$
12. $x^3 - 8x - 3 = 0.$
13. $2x^3 - 5x^2 - 8x - 16 = 0.$
14. $2x^3 + 11x^2 + 16x + 16 = 0.$
15. $6x^3 + 29x^2 + 26x - 21 = 0.$
16. $20x^2 + x - 12 = 0.$
17. $6x^2 - 11x + 5 = 0.$
18. $x^3 + 7x^2 + 14x + 8 = 0.$
19. $x^3 - 7x^2 + 16x - 12 = 0.$
20. $6a^2 + 7ax - 3x^2 = 0.$
21. $3x^2 + 11xa + 6a^2 = 0.$
22. $x^2 + 2xb + b^2 = 0.$
23. $6x^2 - 10xa + 9bx = 15ab.$
24. $35x^2 + x - 88 = 0.$
25. $10x^2 - 19\frac{3}{4}x = -6.$
26. $acx^2 - bc^2x + ax = bc.$
27. $6x^2 = -7\frac{3}{4}x - 2\frac{1}{2}.$
28. $x^3 + 5x^2 - 9x - 45 = 0.$
29. $x^4 - 3x^3 - 14x^2 + 48x = 32.$
30. $x^4 - 11x^2 + 18x - 8 = 0.$
31. $x^4 - 45x^2 - 40x + 84 = 0.$
32. $x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2 = 0.$
33. $x^5 - 2x^4 - 15x^3 + 8x^2 + 68x + 48 = 0.$
34. $x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0.$
35. $x^5 - 3x^4 - 23x^3 + 51x^2 + 94x - 120 = 0.$
36. $x^5 + 3x^4 - 23x^3 - 51x^2 + 94x + 120 = 0.$

Form equations whose roots are:

1. $1, -3, 4$.
2. $5, -6, -3$.
3. $-2, -7, -1$.
4. $1, \frac{2}{3}, \frac{3}{4}$.
5. $-\frac{1}{2}, +3, -\frac{3}{4}$.
6. $\frac{3}{5}, -\frac{7}{2}$.
7. $1\frac{1}{2}, 2\frac{2}{3}$.
8. $-2\frac{3}{4}, -6\frac{5}{7}$.
9. $-5, 7, -3, -1$.
10. $\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}$.
11. $3, -2, -2, -2, 1$.
12. $1, 2, 2, -3, 4$.

SECTION III

COMMON DIVISORS

219. A **Common Divisor** of two or more numbers is a common integral factor of those numbers.

ILLUSTRATION. 3 is a common integral factor of 6, 9, and 21. $\therefore$ it is a common divisor of those numbers.

220. The **Highest Common Divisor (H. C. D.)** of two or more numbers is the product of all the common prime factors of those numbers.

ILLUSTRATION. The prime factors of $18 = 2 \cdot 3 \cdot 3$, of $24 = 2 \cdot 3 \cdot 2 \cdot 2$, of $30 = 2 \cdot 3 \cdot 5$. $\therefore$ the H. C. D. of 18, 24, and 30 is 2×3, or 6.

PRINCIPLES FOR FIRST METHOD

221. The Principles involved in the *First Method* are as follows:

1. A number equals the product of its prime factors.

2. Every prime factor of a number is a divisor of that number.

3. Every product of any two or more prime factors of a number is a divisor of that number.

4. A number has no divisors except its prime factors and the products of every two or more of them. Hence the H. C. D. of two or more numbers is the product of all their common prime factors.

222. PROBLEM 1. **To find the H. C. D. of two or more expressions by factoring.**

Rule. *Resolve the expressions into their prime factors, and take the product of all the common prime factors for the H. C. D.*

Dem. See Principles 1, 2, 3, and 4.

MODEL SOLUTION

Find H. C. D. of the three expressions:

1. $18a^2b(x^6-y^6)(x^3-2x^2+x+x^2y-2xy+y)$.
2. $24axb^2(x^3-6x^2+11x-6)(x^3+x^2y-xy^2-y^3)$.
3. $42a^2(x^3-1)(bx^2-by^2)^2$.

(1) $=2\cdot3\cdot3\cdot a\cdot a\cdot b\cdot(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)(x+y)(x-1)(x-1)$, by Prin. 1.

(2) $=2\cdot2\cdot2\cdot3\cdot a\cdot x\cdot b\cdot b\cdot(x-1)(x-2)(x-3)(x+y)(x+y)(x-y)$, by Prin. 1.

(3) $=2\cdot3\cdot7\cdot a\cdot a\cdot(x-1)\cdot(x^2+x+1)\cdot b\cdot b\cdot(x+y)(x+y)(x-y)(x-y)$, by Prin. 1.

H. C. D. $=2\cdot3\cdot a\cdot(x+y)(x+y)(x-y)$, by Prin. 4.

EXAMPLES

Find H. C. D., and state propositions in factoring used:

1. $x^{12}+a^{12}$, $x^{20}+a^{20}$.
2. $64a^6-x^6y^{12}$, $16a^4-x^4y^8$.
3. $2x^2-x-6$, $2x^3-7x-2$. Check.
4. x^3-x^2-7x+3, x^4+2x^3+2x-1.
5. $x^2+11x+30$, $9x^3+53x^2-9x-18$.
6. $5x^2c-20c$, $3x^5-48x$, $2x^3b-16b$.
7. $2x^2-xy-15y^2$, $x^2-6xy+9y^2$.
8. $4x^3-27x+5$, $6x^2-7x-20$. Check.
9. $24x^4-22x^2+5$, $48x^4+16x^2-15$.
10. $a^2-4ax+4x^2$, a^4-16x^4. Check.
11. x^4-y^4, $x^3-x^2y-xy^2+y^3$. Check.
12. $x^2+(a+b)x+ab$, $x^2+(b+c)x+bc$.
13. $3x^4-14x^3-9x+2$, $2x^4-9x^3-14x+3$.

14. $x^4+2x^2a^2+a^4$, x^4-a^4, $2(x^{10}+a^{10})$.

15. a^3+3a^2+3a+1, $a^4+4a^3+6a^2+4a+1$.

NOTE. Problem 1 is all-sufficient to find the H. C. D. of any algebraic expressions used in elementary work. Another Problem (No. 2) is added, but its use is left to the discretion of the teacher.

PRINCIPLES FOR SECOND METHOD

223. The Principles involved in the *Second Method* are:

1. If the lesser of two numbers is a divisor of the greater, it is their H. C. D.

2. A divisor of a number is a divisor of any multiple of that number.

3. A common divisor of two numbers is a divisor of their sum and of their difference; also of the sum and of the difference of any multiples of them.

4. The H. C. D. of the difference of two numbers and the smaller one of them is the H. C. D. of the two numbers.

Let the student give an illustration of each of these Principles.

224. PROBLEM 2. **To find the H. C. D. of two polynomials by the method of division.**

Rule 1. *Remove all monomial factors from both expressions and write down their common prime factors, if any, as factors of the H. C. D.*

2. *Then divide one polynomial by the other. If the remainder is not zero, divide the divisor by it, and continue the process of dividing the last divisor by the last remainder until the remainder is zero.*

3. *The last divisor multiplied by the common prime factors removed is the H. C. D. required.*

That the demonstration of this rule may be more easily understood, an arithmetical example will be taken first.

Dem. 1. Arithmetical.

Let it be required to find the H. C. D. of 286 and 338.

MODEL SOLUTION

$$\begin{array}{rrrl}
\underline{286}\,| & 338 & \underline{|1} & \\
& \underline{286} & & \\
& \underline{52}\,| & 286 & \underline{|5} \\
& & \underline{260} & \\
\text{H. C. D.} = & & \underline{26}\,| & 52 \ \underline{|2} \\
& & & \underline{52} \\
& & & 0
\end{array}$$

Reasoning. 1. If 286 is a divisor of 338, it is the H. C. D. of the two numbers, by Prin. 1. By trial, 286 is found not to be an exact divisor of 338, for there is a remainder, 52.

2. If 52 is a divisor of 286, it is the H. C. D. of 52 and 286 (Prin. 1). But the H. C. D. of 52 and 286 is the H. C. D. of 286 and 338 (Prin. 4). By trial, 52 is found not to be a divisor of 286, because there is a remainder, 26.

3. If 26 is a divisor of 52, it is the H. C. D. of 26 and 52 (Prin. 1). But the H. C. D. of 26 and 52 is the H. C. D. of 52 and 286 (Prins. 3 and 4). And the H. C. D. of 52 and 286 is the H. C. D. of 286 and 338 as previously shown. By trial, 26 is found to be an exact divisor of 52, and is, therefore, the H. C. D. of 286 and 338.

Dem. 2. Algebraic.

Find the H. C. D. of $18\,abx^4 - 24\,abx^3 + 2\,abx^2 + 4\,abx$ and $24\,a^2x^5 - 4\,a^2x^4 - 20\,a^2x^3 + 8\,a^2x^2$.

(1) $= 2 \cdot a \cdot b \cdot x \cdot (9\,x^3 - 12\,x^2 + x + 2)$.

(2) $= 4 \cdot a^2 \cdot x^2 \cdot (6\,x^3 - x^2 - 5\,x + 2)$.

$\therefore\ 2 \cdot a \cdot x$ are common prime factors belonging to the H. C. D.

$$\begin{array}{llll}
\underline{6x^3-x^2-5x+2}\,| & 9x^3-12x^2+x+2\ \underline{|3} & & \\
& 18x^3-24x^2+2x+4, & \text{by multiplying by 2 to avoid fractions.} & \\
& \underline{18x^3-3x^2-15x+6} & & \\
& \underline{-21x^2+17x-2}\,| & 6x^3-x^2-5x+2\ \underline{|-2x-9} & \\
& & 42x^3-7x^2-35x+14 = \text{1st divisor} \times 7 & \\
& & \underline{42x^3-34x^2+4x} & \\
& & \quad 27x^2-39x+14 & \\
& & \quad 189x^2-273x+98 = \text{last rem.} \times 7 & \\
& & \quad \underline{189x^2-153x+18} & \\
& & \qquad -120x+80 & \\
& & = 3x-2, \text{ by dividing by } -40. &
\end{array}$$

$$\begin{array}{r|l|l} 3x-2 & -21x^2+17x-2 & -7x+1 \\ & \underline{-21x^2+14x} & \\ & \qquad\qquad 3x-2 & \\ & \qquad\qquad \underline{3x-2} & \end{array}$$

$\therefore$ H. C. D. $= 2\,ax\,(3\,x-2)$.

Reasoning

1. Removing the monomial factors $2\,abx$ and $4\,a^2x^2$, the factor (divisor) $2\,ax$ is found to be common, and hence is saved as one of the factors of the H. C. D. Since the remaining monomial factors are not common, they can form no part of the H. C. D., and hence may be rejected.

2. If $6\,x^3-x^2-5\,x+2$ is a divisor of $9\,x^3-12\,x^2+x+2$, it is the H. C. D. of both of them (Prin. 1). To avoid fractions, multiply $9\,x^3-12\,x^2+x+2$ by 2 (Prin. 2). By trial, $6\,x^3-x^2-5\,x+2$ is found not to be an exact divisor of $18\,x^3-24\,x^2+2\,x+4$, for there is a remainder of $-21\,x^2+17\,x-2$.

3. If $-21\,x^2+17\,x-2$ is an exact divisor of $6\,x^3-x^2-5\,x+2$, it is the H. C. D. of $6\,x^3-x^2-5\,x+2$ and $9\,x^3-12\,x^2+x+2$ (Prins. 3 and 4). To avoid fractions, multiply $6\,x^3-x^2-5\,x+2$ by 7 (Prin. 2). By trial, $-21\,x^2+17\,x-2$ is found not to be an exact divisor of $42\,x^3-7\,x^2-35\,x+14$, for there is a remainder of $-120\,x+80$. Since this remainder has a monomial factor, -40, which is not common to the original polynomials, such factor may be rejected. Why?

4. If $3\,x-2$ is an exact divisor of $-21\,x^2+17\,x-2$, it is the H. C. D. of $3\,x-2$ and $-21\,x^2+17\,x-2$ (Prin. 1); and of $-21\,x^2+17\,x-2$ and $6\,x^3-x^2-5\,x+2$ (Prins. 3 and 4); and of $6\,x^3-x^2-5\,x+2$ and $9\,x^3-12\,x^2+x+2$. By trial, $3\,x-2$ is found to be an exact divisor. Hence it is the H. C. D. of $6\,x^3-x^2-5\,x+2$ and $9\,x^3-12\,x^2+x+2$. And $2\,ax\,(3\,x-2)$ is the H. C. D. of $18\,abx^4-24\,abx^3+2\,abx^2+4\,abx$ and $24\,a^2x^5-4\,a^2x^4-20\,a^2x^3+8\,a^2x^2$.

If $a=3$, $b=4$, $x=2$, check H. C. D. $= 2\,ax\,(3\,x-2)$.

Dem. 3. General Case.

Let X and Y represent any two polynomials whose H. C. D. is to be found.

1. Since the H. C. D. consists of all the common factors of X and Y, and of those factors only, all monomial factors may be removed, those common being reserved as factors of the H. C. D., all others rejected.

2. Let A and B be the new polynomials which contain no monomial factors, and let A be the one of a degree $=$ or $< B$. Then their H. C. D. may be found as follows:

$$
\begin{array}{l}
\underline{A}\,|\,B\,\underline{|\,Q} \\
\quad\underline{AQ} \\
\qquad \underline{R}\,|\,A\,|\,\underline{Q'} \\
\qquad\quad \underline{RQ'} \\
\qquad\qquad \underline{R'}\,|\,R\,\underline{|\,Q''} \\
\qquad\qquad\quad \underline{R'Q''} \\
\qquad\qquad\qquad R''
\end{array}
$$

H. C. D. is R' if $R'' = 0$.

3. If A is a divisor of B, it is the H. C. D. of A and B (Prin. 1). In dividing, fractions may be avoided by multiplying the dividend by any required number (Prin. 2). By trial, A is found not to be an exact divisor of B because there is a remainder of R.

4. If R is a divisor of A, it is the H. C. D. of R and A (Prin. 1). But the H. C. D. of R and A is the H. C. D. of A and B (Prins. 3 and 4). By trial, R is found not to be an exact divisor of A because there is a remainder of R'.

5. If R' is a divisor of R, it is the H. C. D. of R' and R (Prin. 1). But the H. C. D. of R' and R is the H. C. D. of R and A (Prins. 3 and 4). And the H. C. D. of R and A is the H. C. D. of A and B as previously shown.

By trial, R' is found to be an exact divisor of R (if $R'' = 0$).

Hence R' is the H. C. D. of A and B.

If $R'' \neq 0$, the process may be repeated until the remainder becomes zero.

Now the common factors of X and Y are found to be R' and the common factors reserved during the operation.

Hence the product of all the common factors is the H. C. D. of X and Y.

225. Cor. The H. C. D. of three or more expressions is obtained by finding the H. C. D. of the first two, and then the H. C. D. of this result and the third, and so on. The last result will be the H. C. D. sought.

Dem. This is evident, since any factor common to three or more expressions must be a factor of any two of them.

Model Solution

Find the H. C. D. of $x^3-9x^2+26x-24$, $x^3-8x^2+5x+14$, and $x^3-18x^2+87x-110$. Use detached coefficients.

$$\begin{array}{r|l}
1-8+\ \ 5+14 & 1-9+26-24 \\
\underline{1-9+26-24} & \underline{1} \\
\end{array}$$

$$\begin{array}{r|l}
1-\ \ 9+26-24 & 1-21+38 \\
\underline{1-21+38\qquad} & \underline{1+1} \\
12-12-24 & \text{or}
\end{array}$$

$$\begin{array}{rl}
1-\ \ 1-\ \ 2, & \text{by rejecting } 12 \\
\underline{1-21+38} & \\
20-40 & \text{or}
\end{array}$$

$$\begin{array}{r|l}
1-21+38 & 1-\ \ 2, \text{ by rejecting } 20 \\
\underline{1-\ \ 2\qquad} & \underline{1-19} \\
-19+38 & \\
\underline{-19+38} &
\end{array}$$

Hence, $x-2$ is the H. C. D. of the first two expressions.

Next find the H. C. D. of $x-2$ and $x^3-18x^2+87x-110$.

$$\begin{array}{r|l}
1-18+87-110 & \underline{-2} \\
\underline{-\ \ 2+32-110} & \\
1-16+55 &
\end{array}$$

Hence, $x-2$ is the H. C. D. of the three expressions.
Check. $x-2$, when $x=4$.

EXAMPLES

Find by Problem 2 the H. C. D. of the following:

1. x^4-y^4, $x^3+x^2y+xy+y^2$.
2. x^3-3x+2, x^3-5x^2+7x-3.
3. x^3+3x^2-4x, $7x^3-18x^2+6x+5$.
4. $4x^4+11x^2+25$, $4x^4-9x^2+30x-25$.
5. $3ax^2-13ax+14a$, $7x^3-17x^2+6x$.
6. $3x^3-13x^2+23x-21$, $15x^3-38x^2-2x+21$.
7. $15x^3+35x^2+3x+7$, $27x^4+63x^3-12x^2-28x$.

8. $x^2 + x - 12,\ x^3 - 5x^2 + 7x - 3.$

9. $x^2 - xa + xb - ab,\ x^3 + x^2b + xa + ba.$

10. $x^4 + x^2a^2 + a^4,\ x^4 + x^3a - xa^3 - a^4.$

11. $4x^5 + 11x^4 + 81,\ 2x^5 - 11x^2 - 9.$

12. $2x^3 + 3x^2 - 2x - 3,\ 3x^4 - x^2 - 2.$

13. $x^3 - 6x^2 + 11x - 6,\ x^3 + 4x^2 + x - 6.$

14. $x^4 + 40x + 39,\ x^5 + 6x^4 - x - 6.$

15. $2x^3 + 5x^2 - 3x,\ x^3 + 2x^2 - 3x.$

16. $x^6 - 6x + 5,\ x^6 + x^5 - 11x + 9.$

17. $x^3 - 7x^2 + 14x,\ 6x^4 - 28x^3 + 16x^2.$

18. $2x^3 - 5x^2 + 7x - 4,\ x^4 - 6x^3 + 6x^2 - 3x + 2.$

19. $12x^5 - 6x^4 - 9x^3 + 3x^2,\quad 18x^5 - 3x^4 - 12x^3 - 3x^2.$

20. $4x^4 - 9x^2 + 6x - 1,\ 6x^3 - 7x^2 + 1.$

21. $3y^4 + 2y^3 - 3y^2 + 2y - 1,\ 3y^5 + 2y^4 + y^2.$

22. $x^4 - x^3 + x^2 - 2x + 1,\ x^5 - x^2 - x + 1.$

23. $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6,\ x^5 + 3x^4 - 8x^2 - 9x - 3.$

24. $8a^4 - 2a^3 - 19a^2 + 3a + 10,\ 6a^4 - a^3 - 3a^2 - 4a - 4.$

25. $x^4 + 10x^3 + 20x^2 - 10x - 21,\ x^4 + 4x^3 - 22x^2 - 4x + 21.$

26. $6x^4 - 13x^3 + 3x^2 + 2x,\ 6x^4 - 9x^3 + 15x^2 - 27x - 9.$

27. $2a^4 - 12a^3 + 19a^2 - 6a + 9,\ 4a^3 - 18a^2 + 19x - 3.$

28. $6x^4 - 5x^3 - 10x^2 + 3x - 10,\ 4x^3 - 4x^2 - 9x + 5.$

29. $4x^4 + 12x^3 - x^2 - 27x - 18,\ 4x^4 + 4x^3 - 17x^2 - 9x + 18.$

30. $x^4 - 3ax^2 - 5b^2x^2 + 15ab^2,\ x^4 - 3ax^2 + 5b^2x^2 - 15ab^2.$

31. $x^3 + y^3 + a^3 - 3xya,\ x(x + 2y) + y(y + 2a) + a(a + 2x).$

32. $52yx^3 - 24yx^4 - 44yx^2 - 12y + 8yx^5 + 60yx,\quad 14y^2z + 60y^2zx^2 - 16y^2zx^3 + 2y^2zx^5 - 74y^2zx - 2y^2zx^4.$

33. $x^3 - 9x^2 + 26x - 24,\ x^4 - 10x^3 + 35x^2 - 50x + 24,\ x^4 - 16.$

34. $x^2 - 42x + 117,\quad x^3 - 49x - 120,\quad x^5 - 3x^4 - 8x^3 + 24x^2 - 9x + 27.$

35. $3x^3 + 9x^2 + 9x + 6,\ 3x^3 + 8x^2 + 5x + 2,\ 2x^4 + 6x^3 + 4x^2.$

SECTION IV

COMMON MULTIPLES

226. A **Common Multiple** of two or more numbers is a number which is exactly divisible by each of them.

227. The **Lowest Common Multiple (L. C. M.)** of two or more algebraic expressions is an expression which contains all the prime factors of each expression once.

228. PRINCIPLE. A multiple of a number must contain all the factors of that number. For, by definition, if the multiple is exactly divisible by a number, it must be divisible by (or contain) all the factors of the number.

229. PROBLEM. **To find the L. C. M. of two or more expressions.**

Rule. *Resolve each expression into its prime factors, and for the L. C. M. multiply the factors of the first expression by all the factors of the second not found in the first, this result by all the factors of the third not found in the first two, and so on.*

Dem. See definition of L. C. M. and Principle.

MODEL SOLUTION

Find the L. C. M. of $6\,mx^3+6\,m$, $(2\,m(x+1))^2$, $9\,x^6-9$.

1. $6\,mx^3+6\,m=2\cdot 3\cdot m(x+1)\cdot(x^2-x+1)$, by Props. 2 and 6.
2. $(2\,m(x+1))^2=2\cdot 2\cdot m\cdot m\cdot(x+1)\cdot(x+1)$, by definition of exponent.
3. $9\,x^6-9=3\cdot 3\cdot(x+1)\cdot(x^2-x+1)\cdot(x-1)\cdot(x^2+x+1)$, by Props. 2, 4, 6.

$$\text{L.C.M.}=2\cdot 3\cdot m\cdot(x+1)\cdot(x^2-x+1)\cdot 2\cdot m\cdot(x+1)\cdot 3\cdot(x-1)\cdot(x^2+x+1)$$

Explanation. The L. C. M. to contain expression (1) must contain all of its factors, *i.e.*, $2\cdot 3\cdot m\cdot(x+1)(x^2-x+1)$. These are placed below the line as part of the L. C. M. The L. C. M. to contain expression (2) must contain all of its factors, *i.e.*, $2\cdot 2\cdot m\cdot m\cdot(x+1)(x+1)$. But

the L. C. M., so far as found, already contains one factor 2, one m, and one $(x+1)$, and hence one more of each needs to be placed in the L. C. M. The L. C. M. to contain expression (3) must contain all of its factors, *i.e.*, $3 \cdot 3(x+1)(x^2-x+1)(x-1)(x^2+x+1)$. But the L. C. M. already contains 3, $(x+1)$, and (x^2-x+1), and hence another 3, also $(x-1)$ and (x^2+x+1) need to be put down. Since the L. C. M., as found, contains all the factors of expressions (1), (2), (3), it is a common multiple of those expressions; and since it contains them but once, it is the L. C. M. of them.

Note. Problem 2 in H. C. D. may be used in finding the factors of expressions difficult of resolution by the usual propositions.

EXAMPLES

Find the L. C. M. and state propositions used:

1. $x^3+64,\ x^2-4x+16.$
2. $(x^3-1)^2,\ (x+1)^2,\ (x^2-1)^2.$
3. $(x^3+1)^2,\ x^2+1,\ (x+1)^3.$
4. $x^2-13x+42,\ x^2+3x-54.$
5. $x^3-x^2-7x+3,\ x^4+2x^3+2x-1.$
6. $2x^4+13a^2x^2+15a^4,\ 3x^5-75a^4x.$
7. $x^3-9x^2+23x-15,\ x^2-8x+7.$
8. $x^8+x^4+1,\ x^2-x+1,\ x^4+x^2+1.$
9. $(x+z)^2-y^2,\ (x+y)^2-z^2,\ (y+z)^2-x^2.$
10. $3x^3+x^2-5x+21,\ 6x^3+29x^2+26x-21.$
11. $2x^4-9x^3-14x+3,\ 3x^4-14x^3-9x+2.$
12. $20x^2+x-12,\ 12x^3-5x^2+5x-6.$
13. $a(b-a)(a-c),\ b(c-b)(a-b),\ c(c-a)(b-c).$
14. $ab-ac-bc+c^2,\ a^2-ac-ab+bc,\ ab-b^2-ac+bc.$
15. $9ac+2a^2-5ab+4c^2+8bc-12b^2,\ ab+2a^2-3b^2-4bc-ac-c^2.$
16. $x^4+2x^3-7x^2+8x-5,\ 2x^4+5x^3-10x^2+14x-15.$

17. $x^4 + x^2a^2 + a^4,\ x^3 + a^3,\ x^3 - a^3.$

18. $x^4 - 7x^2 + 1,\ x^3 - 8x - 3.$

19. $3x^3 - 24x - 9,\ 2x^3 - 16x - 6.$

20. $2x^3 + x^2 - 8x + 5,\ 7x^2 - 12x + 5.$

Find the H. C. D. and the L. C. M. of the following:

1. $x^4 + x^3 + x^2 + x + 1,\ x^5 - 1.$

2. $a^5 - a^2 - a + 1,\ a^4 - a^3 + a^2 - 2a + 1.$

3. $21a^3 + 28a^2 - 15a - 20,\ 21a^3 - 28a^2 - 15a + 20,\ 14a^3 - 21a^2 - 10a + 15.$

4. $6x^4 - x^3 - 3x^2 - 4x - 4,\ 8x^4 - 2x^3 - 19x^2 + 3x + 10.$

5. $x^4 + x^2y^2 + y^4,\ x^2 + xy + y^2,\ x^8 + x^4y^4 + y^8.$

6. $x^3 - 9x^2 + 26x - 24,\ x^4 - 10x^3 + 35x^2 - 50x + 24.$

7. $x^3 - 8x^2 + 5x + 14,\ x^4 - 45x^2 - 40x + 84.$

8. $3x^4 + 13x^3 - 117x - 243,\ x^5 - 3x^4 - 8x^3 + 24x^2 - 9x + 27.$

9. $x^4 - 20x^2 + 64,\ 2x^4 + 23x^3 + 30x^2 - 92x - 152.$

10. $x^3 - 7x^2 + 14x - 8,\ 2x^3 - 5x^2 - 17x + 20.$

11. $x^3 - 3a^2x + 2a^3,\ 2x^3 + ax^2 + a^2x - 4a^3.$

12. $x^2 - 8xa + 15a^2,\ x^2 - 3xa - 10a^2,\ x^2 + 2xa - 35a^2.$

13. $x^4 - x^3 - 4x^2 - x + 1,\ 4x^3 - 3x^2 - 8x - 1.$

14. $12vy + 20xy + 18vz + 30xz,\ 12y^2 + 18yz - 4y - 6z.$

15. $2x^3 - 5x^2 - 8x - 16,\ 2x^3 + 11x^2 + 16x + 16.$

16. $g^2 - 3gh + gi + 2h^2 - 2hi,\ g^2 - h^2 + 2hi - i^2.$

17. $a^{12} + z^{12},\ a^{20} + z^{20},\ a^8 - z^8,\ a^{16} - z^{16},\ a^4 + z^4.$

18. $a^3x + 2a^2x^2 + 2ax^3 + x^4,\ 5a^5 + 10a^4x + 5a^3x^2.$

19. $4a^2d - 4acd - 2ac^2 + 2c^3,\ a^2d^2 - c^2d^2 - a^2c^2 + c^4.$

20. $15x^4 - 2x^3 + 10x^2 - x + 2,\ 9x^5 + 2x^3 + 4x^2 - x + 1.$

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Sample Test Questions

1. Define exponent, power, coefficient, multiplication, symmetry.
2. When may multiplication by detached coefficients be performed?
3. How many kinds of terms are there in the square of a binomial? Of a polynomial?
4. How many kinds of terms are there in the cube of a binomial? Of a polynomial?
5. State the law for the product of $(x+3)(x-2)(x+5)$.
6. State the Remainder Theorem, and give its demonstration.
7. State the Factor Theorem, and give its demonstration.
8. State the Division Theorem, and give its demonstration.
9. Define factor, composite number, prime number, H. C. D., L. C. M.
10. State the Special Theorems upon which the following propositions depend: Prop. 3, Prop. 4, Prop. 5, Prop. 6, Prop. 8.
11. State the propositions used in factoring each of the following expressions: $x^5 - y^5$, $x^4 + 2x^2 + 1$, $8x^3 + 1$, $1 - 27y^3$, $6a^2x^3y^5$, $6x^2 - 5x - 6$, $x^4 + x^2y^2 + y^4$, $5ax - 5a - 3bx + 3b$, $x^{12} - y^{12}$.
12. State Propositions 7 and 9, and tell their uses.
13. State the method of solving an equation of the form of $(x-2)(x+3)(x-7)=0$. State the proposition governing the solution. Is it right to divide both members by $x - 2$? Why not? Find the *error* in the following: If x and $a = 5$, then $10 = 0$. For $x + a = x + a$, $x^2 + ax = ax + a^2$, $x^2 - a^2 = 0$, $x + a = 0$, by dividing both members by $x - a$. But $x + a$ is $5 + 5$, or 10. $\therefore 10 = 0$.
14. Having the roots of an equation, how is the equation formed?
15. How is the H. C. D. found by factoring? State principles used.
16. How is the L. C. M. found by factoring? State principle used.

CHAPTER VI

FRACTIONAL EXPRESSIONS

SECTION I

CLASSIFICATION AND DEFINITIONS

230. A **Fraction** is an indicated operation in division, the numerator corresponding to the dividend, and the denominator to the divisor, as $\frac{a}{b}$.

231. The **Terms** of a fraction are the numerator and the denominator.

232. The **Value** of a fraction is the quotient of the numerator divided by the denominator. Since the divisor multiplied by the quotient equals the dividend, $b \times \frac{a}{b} = a$.

233. The **Reciprocal** of a number is one divided by that number, or is the number inverted. Thus the reciprocal of 3 is $\frac{1}{3}$; of $\frac{a}{b}$ is $\frac{b}{a}$.

DIAGRAM OF FRACTIONS

- FRACTIONS
  - 1. Common
    - 1. Simple
      - 1. Proper
      - 2. Improper
        - 1. Degree of numerator = degree of denominator.
        - 2. Degree of numerator > degree of denominator.
    - 2. Compound
    - 3. Complex.
  - 2. Decimal
  - 3. Duodecimal.

234. A **Common Fraction** is a fraction in which both terms are expressed, one above and the other below a line.

235. A **Decimal Fraction** is a fraction whose denominator is 10, 100, 1000, or etc. Ex. $\frac{6}{10}$, .07.

236. A **Simple Fraction** is a fraction in which both terms are in the integral form.

ILLUSTRATIONS. $\dfrac{3a^3+m^2}{5x-7d}$ and $\dfrac{5amx^4}{x-2y+3z}$.

237. A **Compound Fraction** is a fraction composed of two or more simple fractions connected by the sign of multiplication.

ILLUSTRATION. $\dfrac{a}{b}\times\dfrac{m}{n}\times\dfrac{x}{y}$.

238. A **Complex Fraction** is a fraction having in one or both of its terms an expression in the fractional form.

ILLUSTRATIONS. $\dfrac{\frac{3}{4}ax+\dfrac{x}{y}}{\dfrac{cd}{8}}$, $\dfrac{2\frac{5}{7}}{3}$, $\dfrac{1}{2-\dfrac{3}{4-\frac{5}{6}}}$.

239. A **Continued Fraction** is a complex fraction of the form

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\cfrac{g}{h+\cdots}}}}$$

and may have a finite or an infinite number of links in the chain of operations. In the former case it is *terminating*, in the latter, *non-terminating*.

240. A **Proper Fraction** in the literal notation is one in which the degree of the numerator is less than the degree of the denominator.

ILLUSTRATIONS. $\frac{1}{x}$ and $\frac{x}{x^2-1}$.

241. An **Improper Fraction** in the literal notation is a fraction in which the degree of the numerator is equal to or greater than the degree of the denominator.

ILLUSTRATIONS. $\frac{x}{x-1}$, $\frac{x+1}{x+2}$, $\frac{x^2+1}{x+3}$.

242. The **Sign** of a fraction is the sign + or − written before the line separating the numerator and the denominator. It is customary for the first term in the numerator and in the denominator to have the sign + understood, and when any fraction is not so written it should be changed accordingly.

SECTION II

REDUCTION

243. **Reduction** of fractions is the process of changing the form without changing the value.

244. A fraction is said to be in its **Lowest Terms** when the numerator and the denominator contain no common integral factor.

245. PROBLEM 1. **To reduce a fraction to its lowest terms.**

Rule 1. *Resolve both numerator and denominator into their prime factors and cancel all common factors; or*

Rule 2. *Divide both terms by their H. C. D.*

Dem. That these two processes reduce the fraction to its lowest terms is true by definition (Art. **244**). That the value of the fraction is not changed is true in accordance with the following Principle:

246. PRINCIPLE 1. Dividing or multiplying both terms of a fraction by the same number does not change the value of the fraction.

Dem. 1. By means of an equation.

Let $\frac{a}{b}$ represent a fraction, and x the number by which both terms of the fraction are to be divided.

Then $$\frac{a}{b} \equiv \frac{a \div x}{b \div x},$$

for $$ab \div x \equiv ab \div x, \text{ by clearing of fractions.}$$

$\therefore$ dividing both terms of a fraction by the same number does not change its value.

Let the student prove the case of multiplying both terms of a fraction by the same number.

Dem. 2. By means of the principles of elementary arithmetic.

The numerator corresponds to the dividend, the denominator to the divisor, the value of the fraction to the quotient. If the dividend and the divisor are divided by the same number, the value of the quotient is not changed. But the quotient is the value of the fraction. $\therefore$ the value of the fraction is not changed.

247. PRINCIPLE 2. Any two of the three signs in a fraction, *i.e.*, the sign of the fraction, the sign of the numerator, and the sign of the denominator, may be changed without affecting the value of the fraction.

Let the student prove this Principle by stating the reasons for the steps in the following possible cases:

1. $+\dfrac{+a}{+b} \equiv +\dfrac{a}{b}.$

2. $+\dfrac{-a}{+b} \equiv -\dfrac{a}{b}.$

3. $+\dfrac{+a}{-b} \equiv -\dfrac{a}{b}.$

4. $+\dfrac{-a}{-b} \equiv +\dfrac{a}{b}.$

5. $-\dfrac{+a-x}{+b+y} \equiv -\dfrac{a-x}{b+y}.$

6. $-\dfrac{-a+x}{+b+y} \equiv +\dfrac{a-x}{b+y}.$

7. $-\dfrac{+a-x}{-b-y} \equiv +\dfrac{a-x}{b+y}.$

8. $-\dfrac{-a-x}{-b+y} \equiv -\dfrac{a+x}{b-y}.$

Cyclic Order

$$\frac{x}{(a-b)(c-b)} - \frac{y}{(c-b)(c-a)} - \frac{z}{(a-c)(b-a)}$$

$$\equiv -\frac{x}{(a-b)(b-c)} + \frac{y}{(b-c)(c-a)} - \frac{z}{(c-a)(a-b)}.$$

Let the student state the changes made in each fraction and the reasons why the value is not changed.

EXAMPLES

Arrange the letters of the denominators in cyclic order:

1. $\dfrac{1}{a(a-b)(a-c)} - \dfrac{1}{b(b-c)(b-a)} + \dfrac{1}{c(c-a)(c-b)}.$

2. $\dfrac{a+b}{(c-b)(a-c)} - \dfrac{b+c}{(a-b)(a-c)} + \dfrac{c+a}{(b-a)(c-b)}.$

Find the real signs of the following fractions, if $a=2$, $b=-3$, $c=4$, $x=5$, $y=6$, $z=-7$.

1. $-\dfrac{-2a^2bc^3x^2y}{+3bxy^2z}.$

2. $-\dfrac{-5b^2za^2c^3}{-4xc^2b^4a^3}.$

3. $+\dfrac{+bcxyz}{-ax^2z^3}.$

4. $+\dfrac{+abc}{+xyz}.$

5. $-\dfrac{+2a^2-3b^2+z}{-3(x-y+c^2)}.$

6. $+\dfrac{(a-b)(y-z)}{(c-x)(b+z)}.$

MENTAL EXERCISES

Reduce to the lowest terms:

1. $\dfrac{21\,a^2m^3x^4}{91\,a^3mx^4}$.

2. $\dfrac{x^2-y^2}{x^2+2\,xy+y^2}$.

3. $\dfrac{1+x+x^2}{1-x^3}$.

4. $\dfrac{x^2+x+1}{x^4+x^2+1}$.

5. $\dfrac{x^3-1}{x^4+x^2+1}$.

6. $\dfrac{x^4-a^4}{x^6+a^6}$.

7. $\dfrac{x^{12}+y^{12}}{x^{20}+y^{20}}$.

8. $\dfrac{x^3+1}{x^5+1}$.

9. $\dfrac{a^4-b^4}{a^4+2\,a^2b^2+b^4}$.

10. $\dfrac{x^2-5\,x+6}{x^2-9}$.

11. $\dfrac{3-a}{a^2-9}$.

12. $\dfrac{x^2-4\,a^2}{x^3+8\,a^3}$.

13. $\dfrac{x^2-10\,x+9}{x^2+5\,x-6}$.

14. $\dfrac{3\,x^2-9\,x+15}{4\,x^2-12\,x+20}$.

15. $\dfrac{x^2+(a-b)\,x-ab}{x^2-(c-a)\,x-ac}$.

EXAMPLES

1. $\dfrac{x^3+x^2+x+1}{x^4-1}$.

2. $\dfrac{x^2-(y-z)^2}{(x+z)^2-y^2}$.

3. $\dfrac{6\,x^2-5\,x-6}{10\,x^2-3\,x-18}$.

4. $\dfrac{a^4+a^2b^2+b^4}{a^4+a^3b-ab^3-b^4}$.

5. $\dfrac{ax+b^2y^2}{a^3x^5+b^{10}y^{10}}$.

6. $\dfrac{x^3-57\,x-56}{x^3+x^2-64\,x-64}$.

7. $\dfrac{4\,a^3-23\,a+5}{10\,a^3-23\,a^2+1}$.

8. $\dfrac{a^3+2\,a^2-5\,a-6}{a^3-3\,a^2-10\,a+24}$.

9. $\dfrac{acx^2+(ad+bc)\,x+bd}{abx^2+(ac+b^2)\,x+bc}$.

10. $\dfrac{x^5+3\,b^4x-4\,x^3b^2}{x^4+bx^3-b^3x-b^4}$.

11. $\dfrac{x^3-7\,x^2+16\,x-12}{3\,x^3-14\,x^2+16\,x}$.

12. $\dfrac{x^8+x^6a^2+x^2a+a^3}{x^4-a^4}$.

13. $\dfrac{x^3-3\,x^2+3\,x-2}{x^3-4\,x^2+6\,x-4}$.

14. $\dfrac{2\,xy^3+xy^2-8\,xy+5\,x}{7\,y^3-12\,y^2+5\,y}$.

15. $\dfrac{x^3-2\,x^2y+2\,xy^2-y^3}{x^4+x^2y^2+y^4}$.

16. $\dfrac{6x^2 + 7xy - 3y^2}{6x^2 + 11xy + 3y^2}$.

17. $\dfrac{x^2 + (a+b)x + ab}{x^2 + (b+c)x + bc}$.

18. $\dfrac{a^3 + b^3 + c^3 - 3abc}{(a-b)^2 + (b-c)^2 + (c-a)^2}$.

19. $\dfrac{(a+b)^5 - a^5 - b^5}{(a+b)^4 - a^4 + b^4}$.

20. $\dfrac{z^m - mz^m + m - 1}{mz^{2m} - (m+1)z^m + 1}$.

21. $\dfrac{15a^4 - 2a^3 + 10a^2 - a + 2}{9a^5 + 2a^3 + 4a^2 - a + 1}$.

248. Problem 2. **To reduce an improper fraction to a whole or a mixed number.**

Rule. *Divide the numerator by the denominator.*

Dem. See the definition of a fraction.

Model Solution

Reduce $\dfrac{x^3 - 2x^2}{x^2 - x + 1}$ to a mixed form.

$$\begin{array}{r|l}
x^3 - 2x^2 \qquad & x^2 - x + 1 \\
\cline{2-2}
\underline{x^3 - x^2 + x} \qquad & x - 1 - \dfrac{2x - 1}{x^2 - x + 1} = \text{mixed form.} \\
-x^2 - x \qquad & \\
\underline{-x^2 + x - 1} & \\
-2x + 1 = \text{remainder.} &
\end{array}$$

Questions. 1. What is done with the remainder? 2. What sign is placed before the fractional form of the quotient? Why? 3. Why were the signs of the remainder changed when it was placed in the numerator? Are they always changed? 4. Is the sign before the fractional form the same as the one before the first term in the remainder? If not, what became of the latter sign? 5. How may the remainder be written so that the sign of the fraction in the quotient shall be +?

EXAMPLES

Reduce to a mixed form:

1. $\dfrac{a^5x^5}{a^2 - x^2}$.

2. $\dfrac{6x^2y + 2y^3}{x + y}$.

3. $\dfrac{x^3 - ax^2 + ax - 2a^2}{a - x}$.

4. $\dfrac{3ax - 3ay - 2a - y^2}{x - y}$.

5. $\dfrac{x^3-x+7}{x^2+x+1}$.

6. $\dfrac{2x^4-x^3+7}{x^2-x+1}$.

7. $\dfrac{3x^3-7x^2+5}{3x^2-4x+4}$.

8. $\dfrac{2ab^3-bc^2+3abc^2-a^3}{b^2-bc}$.

9. $\dfrac{3a^2x-ay^2-2xy^2+am+mx}{a+x}$.

10. $\dfrac{x^7-y^7}{x^2+y^2}$.

11. $\dfrac{x^4+y^4}{x+y}$.

12. $\dfrac{x^3-y^3}{x+y}$.

13. $\dfrac{x^4+y^4}{x^2-xy+y^2}$.

14. $\dfrac{a^2+2ab+b^2+a^3-b^3}{a+b}$.

15. $\dfrac{(a+b)^2(c^2-d^2)-(a^2-b^2)(c+d)^2}{(a+b)(c-d)-(a-b)(c+d)}$.

249. PROBLEM 3. **To reduce fractions to equivalent fractions having the Lowest Common Denominator (L. C. D.).**

Rule. *Find the L. C. M. of the denominators, and multiply both terms of each fraction by the quotient of the L. C. M. divided by its denominator.*

Dem. The L. C. M. of the denominators must be the L. C. D. Why? The L. C. M. is divided by each denominator to get a number by which to multiply the old denominator to make the new denominator. The numerator is multiplied by the same number so as not to change the value of the fraction.

MODEL SOLUTION

Reduce $\dfrac{3}{x^2+5x+6}$, $\dfrac{2x-1}{4-x^2}$, $\dfrac{a}{20+6x-2x^2}$ to L. C. D.

$\dfrac{3}{x^2+5x+6}$, $-\dfrac{2x-1}{x^2-4}$, $-\dfrac{a}{2x^2-6x-20}$, denominators arranged.

$\dfrac{3}{(x+2)(x+3)}$, $-\dfrac{2x-1}{(x+2)(x-2)}$, $-\dfrac{a}{2(x-5)(x+2)}$, denominators factored.

$$\text{L. C. D.} = (x+2)(x+3)(x-2)\cdot 2\cdot(x-5)$$

$$\frac{3\cdot 2(x-2)(x-5)}{(x+2)(x+3)\cdot 2(x-2)(x-5)} = \frac{6x^2-42x+60}{\text{L. C. D.}}$$

$$-\frac{2x-1\cdot 2(x+3)(x-5)}{(x+2)(x-2)\cdot 2(x+3)(x-5)} = -\frac{4x^3-10x^2-56x+30}{\text{L. C. D.}}$$

$$-\frac{a\cdot(x+3)(x-2)}{2(x-5)(x+2)\cdot(x+3)(x-2)} = -\frac{ax^2+ax-6a}{\text{L. C. D.}}.$$

EXAMPLES

Reduce to the L. C. D.:

1. $9y,\ \frac{1+2y}{8},\ \frac{1}{9y},\ 3x,\ m,\ \frac{x+y}{x^2-y^2}.$

2. $\frac{7}{x^2-x-12},\ \frac{8}{x^2+x-6},\ \frac{9}{x^2+x-20}.$

3. $\frac{a}{x^4+x^2y^2+y^4},\ \frac{b}{x^2+xy+y^2},\ \frac{c}{x^2-xy+y^2}.$

4. $\frac{x-1}{x^2+4x+3},\ \frac{x-2}{x^2-4x-5},\ \frac{x-3}{x^2-8x+15}.$

5. $\frac{3}{(x+z)^2-y^2},\ \frac{4}{(x+y)^2-z^2},\ \frac{5}{(y+z)^2-x^2}.$

6. $\frac{s}{a(a-b)(a-c)},\ \frac{s}{b(b-c)(b-a)},\ \frac{s}{c(c-a)(c-b)}.$

SECTION III

ADDITION AND SUBTRACTION

250. PROBLEM. **To add and subtract fractions.**

Rule. *Reduce the fractions, if necessary, to forms having the L. C. D.; then combine the numerators as indicated by the signs + and − before each fraction, and write the result over the L. C. D.*

Dem. See Distributive Law of division.

1. $\frac{a}{x}+\frac{b}{x}-\frac{c}{x}=\frac{a+b-c}{x}$, a general result.

2. $\frac{a}{r}+\frac{b}{s}-\frac{c}{t}=\frac{ast}{rst}+\frac{brt}{rst}-\frac{crs}{rst}=\frac{ast+brt-crs}{rst}$, a particular result.

MODEL SOLUTION

Combine $\frac{-2xy}{x^2-y^2}-\frac{x^2y-xy^2+y^3}{x^3+y^3}-\frac{y}{y-x}$.

1. $\frac{-2xy}{(x+y)(x-y)}-\frac{y(x^2-xy+y^2)}{(x+y)(x^2-xy+y^2)}+\frac{y}{x-y}$, by factoring and arranging.

2. $\frac{-2xy}{(x+y)(x-y)}-\frac{y}{x+y}+\frac{y}{x-y}$, by reducing to lowest terms.

3. $\frac{-2xy}{(x+y)(x-y)}-\frac{xy-y^2}{(x+y)(x-y)}+\frac{xy+y^2}{(x+y)(x-y)}$, by reducing to L. C. D.

4. $\frac{-2xy-(xy-y^2)+(xy+y^2)}{(x+y)(x-y)}$, by Distributive Law.

5. $\frac{-2xy-xy+y^2+xy+y^2}{(x+y)(x-y)}$, by removing ( ).

6. $\frac{-2xy+2y^2}{(x+y)(x-y)}$, by combining terms.

7. $\frac{-2y(x-y)}{(x+y)(x-y)}$, by factoring.

8. $\frac{-2y}{x+y}$, by reducing.

9. $-\frac{2y}{x+y}$, by changing signs.

251. From this example it may be seen that —

1. The denominators should be arranged according to the same letter, as in (1).

2. All fractions should be reduced to their lowest terms before being reduced to the L. C. D., as in (2).

3. It is unnecessary to multiply together the factors of the L. C. D., and the indicated multiplication is more convenient for the purpose of division, as in (3), (7).

4. The dividing line of a fraction is a symbol of aggregation, and must be treated as such. (3), (4), (5), (9).

5. It is customary to have the first term of the numerator positive, as shown in (9).

252. Cor. 1. Integral and mixed forms of expression are changed to the fractional form by multiplying the integral part by the given denominator, annexing the result to the numerator of the fractional part (if any), and writing the result over the given denominator.

Dem. This is but a special case of reducing to the L.C.D. and combining.

Model Solutions

1. Reduce $x^4 - x^2a^2 + a^4$ to a fractional form with the denominator $x^2 + a^2$.

$$(x^4 - x^2a^2 + a^4)\cdot\frac{x^2+a^2}{x^2+a^2} = \frac{(x^4 - x^2a^2 + a^4)(x^2+a^2)}{x^2+a^2} = \frac{x^6+a^6}{x^2+a^2}.$$

2. Reduce $x - 1 - \dfrac{x^3-3}{x^2+x+1}$ to a fractional form.

$$x - 1 - \frac{x^3-3}{x^2+x+1} = \frac{(x-1)(x^2+x+1)}{x^2+x+1} - \frac{x^3-3}{x^2+x+1}$$

$$= \frac{x^3-1}{x^2+x+1} - \frac{x^3-3}{x^2+x+1} = \frac{2}{x^2+x+1}.$$

253. Cor. 2. Mixed forms of expression are added or subtracted in two ways: (1) By adding or subtracting the integral and fractional parts separately and combining these results. (2) By reducing them to the fractional form and adding or subtracting in the usual way.

Model Solutions

Add $x+5-\dfrac{2x}{x-3}$ and $x-2+\dfrac{4}{x+2}$.

1. $(x+5)+(x-2)=2x+3.$

$$-\frac{2x}{x-3}+\frac{4}{x+2}=-\frac{2x^2+4x}{(x-3)(x+2)}+\frac{4x-12}{(x-3)(x+2)}$$

$$=\frac{-2x^2-12}{(x-3)(x+2)}=-2-\frac{2x+24}{(x-3)(x+2)}.$$

$$2x+3-2-\frac{2x+24}{(x-3)(x+2)}=2x+1-\frac{2(x+12)}{(x-3)(x+2)}, \text{ the sum.}$$

2. $$x+5-\frac{2x}{x-3}=\frac{x^2+2x-15-2x}{x-3}=\frac{x^2-15}{x-3}.$$

$$x-2+\frac{4}{x+2}=\frac{x^2-4+4}{x+2}=\frac{x^2}{x+2}.$$

$$\frac{x^2-15}{x-3}+\frac{x^2}{x+2}=\frac{(x^3+2x^2-15x-30)+(x^3-3x^2)}{(x-3)(x+2)}$$

$$=\frac{2x^3-x^2-15x-30}{x^2-x-6}$$

$$=2x+1-\frac{2(x+12)}{x^2-x-6}, \text{ the sum.}$$

EXAMPLES

Combine and reduce to simplest form:

1. $\dfrac{a}{2}+\dfrac{a}{3}+\dfrac{a}{6}+a.$ Check.

2. $\dfrac{x+y}{2}+\dfrac{x-y}{2}-3x+3.$ Check.

3. $\dfrac{x}{x+y}+\dfrac{y}{x-y}-2x-y.$

4. $2x+\dfrac{x-2}{3}+3x+\dfrac{2x-3}{4}.$

5. $\dfrac{7x^2-9x+11}{10}-\dfrac{9x^2-11x+15}{20}.$

6. $\dfrac{a-b}{ab}+\dfrac{b-c}{bc}+\dfrac{c-a}{ca}$.

7. $\dfrac{ax}{a^2-x^2}-\dfrac{a-x}{a+x}+x$.

8. $\dfrac{4x}{5}-\dfrac{3x+1}{x+1}-(3x-5)$.

9. $\dfrac{a^2-ab+b^2}{a-b}+\dfrac{a^2+ab+b^2}{a+b}$.

10. $\dfrac{13a-5b}{4}-\dfrac{3a}{5}-\dfrac{7a-2b}{6}$.

11. $\dfrac{4x^2-11x-5}{5x+5}+\dfrac{1+3x}{1+x}-4x$.

12. $3+\dfrac{2a}{5}+5-\dfrac{3a-2x}{x}+7+\dfrac{x-a}{a}$.

13. $\dfrac{1}{x+3}+\dfrac{x+1}{x^2-3x+9}+\dfrac{x^2+x+1}{x^3+27}$.

14. $\dfrac{2}{x}+\dfrac{3}{1-2x}-\dfrac{2x-3}{4x^2-1}$. Check.

15. $\left(4x-\dfrac{2x-3}{5}\right)-\left(5x+\dfrac{x-2}{3}\right)$.

16. $\dfrac{2}{x^3+x^2+x+1}+\dfrac{3}{x^3-x^2+x-1}$.

17. $\dfrac{6}{1+2x}-\dfrac{3(1-2x)}{4x^2-1}-\dfrac{5}{2x-1}$.

18. $\dfrac{1}{4a^3(a+x)}+\dfrac{1}{4a^3(a-x)}+\dfrac{1}{2a^2(a^2+x^2)}$.

19. $\dfrac{a-b}{a+b}+\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$.

20. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}+\dfrac{1}{c(c-a)(c-b)}$.

21. $\dfrac{a}{bc}+\dfrac{b}{ca}+\dfrac{c}{ab}-2\left(\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}\right).$

22. $\dfrac{x^2+xy+y^2}{x^3-y^3}-\dfrac{x^2-xy+y^2}{x^3+y^3}.$

23. $\dfrac{a+c}{(a-b)(x-a)}-\dfrac{b+c}{(a-b)(x-b)}.$

24. $\dfrac{1}{x-a}-\dfrac{1}{a+x}-\dfrac{2a}{a^2+x^2}-\dfrac{4a^3}{x^4-a^4}.$

25. $\dfrac{9}{2x+6}-\dfrac{1}{2x+2}-\dfrac{4}{x+2}-\dfrac{x-1}{x^2+5x+6}.$

26. $\dfrac{2x+14}{x^3+x^2-49x-49}-\dfrac{1-x}{x^2-6x-7}.$

27. $\dfrac{1}{1-x}-\dfrac{1}{1+x}-\dfrac{2x}{1+x^2}-\dfrac{4x^3}{x^4+1}+\dfrac{8x^7}{x^8-1}.$

28. $\dfrac{2+5x}{(x-1)(3-x)}-\dfrac{2x+3}{(2-x)(x-1)}+\dfrac{3x-2}{x^2-5x+6}.$

29. $\dfrac{x+2}{x^2+x-2}+\dfrac{x+3}{x^2+4x+3}+\dfrac{2x+8}{x^3+4x^2-x-4}.$

30. $\dfrac{x^2-(y-z)^2}{(x+z)^2-y^2}+\dfrac{y^2-(x-z)^2}{(x+y)^2-z^2}+\dfrac{z^2-(x-y)^2}{(y+z)^2-x^2}.$

31. $\dfrac{x}{x^2+5x+6}+\dfrac{15}{x^2+9x+14}-\dfrac{12}{x^2+10x+21}.$

Section IV

MULTIPLICATION

254. Problem 1. **To multiply a fraction by an integral expression.**

Rule. *Multiply the numerator, or divide the denominator, by the integer.*

Dem. 1. Let $\frac{r}{s}$ represent the fraction, and a the integral expression, and let $x =$ the product.

Then $$x = \frac{ar}{s}, \text{ or } \frac{r}{s \div a}.$$

For 1. $x = \frac{r}{s} \cdot a$, by assumption.

2. $sx = s \cdot \frac{r}{s} \cdot a = r \cdot a$. Why? See Art. **232.**

3. $x = \frac{ar}{s}$. Why? State Axiom. Or

4. $x = \frac{r}{s \div a}$, by dividing both terms of $\frac{ar}{s}$ in (3) by a.

Dem. 2. The numerator corresponds to the dividend, the denominator to the divisor, and the value of the fraction to the quotient. If the dividend be multiplied, or the divisor be divided, by an integer, the quotient is multiplied. But the quotient is the value of the fraction. Hence the fraction is multiplied.

Model Solutions

1. Multiply $\frac{x}{a^2 + b^2}$ by $a + b$.

$$\frac{x}{a^2 + b^2} \times (a + b) = \frac{x(a + b)}{a^2 + b^2},$$ by multiplying the numerator.

2. Multiply $\frac{x}{91(a^6 + b^6)}$ by $13(a^2 + b^2)$.

$$\frac{x}{91(a^6 + b^6)} \times 13(a^2 + b^2) = \frac{x}{7(a^4 - a^2b^2 + b^4)},$$ by dividing the denominator.

3. Multiply $\dfrac{x}{15(a^2-ab+b^2)y}$ by $12(a^4+a^2b^2+b^4)$.

$$\frac{x}{15(a^2-ab+b^2)y}\times 12(a^4+a^2b^2+b^4)$$

$$\equiv\frac{x}{3\cdot 5(a^2-ab+b^2)y}\cdot 3\cdot 4\cdot(a^2-ab+b^2)(a^2+ab+b^2)$$

$=\dfrac{x}{5y}\cdot 4\cdot(a^2+ab+b^2)$, by dividing the denominator by $3\cdot(a^2-ab+b^2)$,

$=\dfrac{4x(a^2+ab+b^2)}{5y}$, by multiplying the numerator by $4\cdot(a^2+ab+b^2)$.

Questions. 1. In solution (1), why are ( ) put around the multiplier $a+b$ and not around it in the example? 2. Why not divide the denominator by $a+b$? 3. In solution (2), explain why a^6+b^6 is exactly divisible by a^2+b^2; state remainder test; state Remainder Theorem. 4. In solution (3), is either method of division used or are both methods used? 5. How is the fraction multiplied by a^2-ab+b^2? 6. How by a^2+ab+b^2? 7. How is $a^4+a^2b^2+b^4$ factored?

255. PROBLEM 2. **To multiply by a fraction.**

Rule 1. *Multiply the multiplicand by the numerator and divide the result by the denominator; or*

Rule 2. *Divide by the denominator and multiply the result by the numerator; or*

Rule 3. *Reduce the multiplicand, if necessary, to the fractional form and multiply the numerators together for a new numerator and the denominators together for a new denominator.*

Dem. 1. Let it be required to multiply $\frac{r}{s}$ by $\frac{a}{b}$, $\frac{r}{s}$ being an integer or a fraction, depending on the values of r and s.

Let $\qquad x=$ the product.

Then $\qquad x=\dfrac{ra}{sb}$, or $\dfrac{r\div b}{s\div a}$.

For 1. $x = \frac{r}{s} \cdot \frac{a}{b}$, by assumption.

2. $sbx = r \cdot a$, by clearing of fractions. State Axiom.

3. $x = \frac{ra}{sb}$, by dividing by sb. State Axiom. Or

4. $x = \frac{r \div b}{s \div a}$, by dividing both terms of fraction in (3) by ab.

Hence $\frac{r}{s} \cdot \frac{a}{b} = \frac{ra}{sb}$, or $\frac{r \div b}{s \div a}$. State conclusion in words.

Dem. 2. When the multiplicand is multiplied by the numerator, it is multiplied by a number that is the denominator times too large. To make this product right, divide it by the denominator of the multiplier.

Let the student state the reasons for Rules (2) and (3).

Model Solutions

1. Multiply $a^{12} + b^{12}$ by $\frac{5}{a^4 + b^4}$.

$$
\begin{aligned}
(a^{12} + b^{12}) \times \frac{5}{a^4 + b^4} &= (a^{12} + b^{12}) \times 5 \div (a^4 + b^4) \\
&= 5(a^{12} + b^{12}) \div (a^4 + b^4) \\
&= 5(a^8 - a^4b^4 + b^8).
\end{aligned}
$$

$$
\begin{aligned}
\text{Or, } (a^{12} + b^{12}) \times 5 \div (a^4 + b^4) &= (a^{12} + b^{12}) \div (a^4 + b^4) \times 5 \\
&= (a^8 - a^4b^4 + b^8) \times 5 \\
&= 5(a^8 - a^4b^4 + b^8).
\end{aligned}
$$

2. Multiply $\frac{2\,ax^3 - 2\,a^3x}{x^2 + 2\,xa + a^2}$ by $\frac{x^3 + a^3}{x^3 - 2\,x^2a + 2\,xa - a^3}$.

$$
\frac{2\,ax\cancel{(x + a)}\cancel{(x - a)}}{\cancel{(x + a)}(x + a)} \times \frac{\cancel{(x + a)}\cancel{(x^2 - xa + a^2)}}{\cancel{(x - a)}\cancel{(x^2 - xa + a^2)}}, \text{ by factoring}
$$

$= 2\,ax$, by cancellation.

3. $\frac{x-5}{5x+6} \times \frac{x+5}{3x-8} = \frac{x^2-25}{15x^2-22x-48}$, by multiplying together the numerators for a new numerator and the denominators for a new denominator.

4. Multiply together

$$\frac{x^2+3x-10}{2x^2-21-x}, \frac{3x^2+4x-15}{14x^2+27x+9}, \frac{14x^2-43x-21}{5x^2+19x-30}.$$

$$\frac{(x-2)\cancel{(x+5)}}{\cancel{(x+3)}\cancel{(2x-7)}} \times \frac{(3x-5)\cancel{(x+3)}}{\cancel{(7x+3)}(2x+3)} \times \frac{\cancel{(2x-7)}\cancel{(7x+3)}}{\cancel{(x+5)}(5x-6)}$$
$$= \frac{(x-2)(3x-5)}{(2x+3)(5x-6)} = \frac{3x^2-11x+10}{10x^2+3x-18}.$$

Questions. 1. In solution (1), why is it better to divide by the denominator and then multiply by the numerator? 2. How may a^4+b^4 and $a^8-a^4b^4+b^8$ be factored? 3. In solution (2), state propositions used in factoring. 4. Explain each step in the process of cancellation, and show that each step is a process of multiplication or division. 5. In solution (3), why multiply $x-5$ by $x+5$? 6. Why $5x+6$ by $3x-8$? 7. In the final result, why may not the x^2 in the numerator and the x^2 in the denominator be canceled? What is cancellation?

256. COR. 1. Any number of fractions may be multiplied together by the method of cancellation, or by multiplying the numerators together for a new numerator, and the denominators together for a new denominator.

Dem. This is a direct consequence of Problem 2, as may be shown by multiplying two fractions together and this product by a third, etc. See model solution No. 4.

257. COR. 2. Mixed forms of expression may be multiplied together by reducing them to fractional forms and then multiplying in the usual way.

258. COR. 3. Fractional polynomials may be multiplied together in the same manner as polynomials in the integral form.

Model Solution

Multiply $\frac{a^2}{2}-\frac{a}{3}+\frac{1}{4}$ by $\frac{2a}{3}-\frac{1}{2}$.

$$
\begin{array}{rrrr}
\frac{a^2}{2} & -\frac{a}{3} & +\frac{1}{4} & \\
\frac{2a}{3} & -\frac{1}{2} & & \\
\hline
\frac{a^3}{3} & -\frac{2a^2}{9} & +\frac{a}{6} & \\
 & -\frac{a^2}{4} & +\frac{a}{6} & -\frac{1}{8} \\
\hline
\frac{a^3}{3} & -\frac{17a^2}{36} & +\frac{a}{3} & -\frac{1}{8}.
\end{array}
$$

EXAMPLES

1. $\dfrac{x^2-a^2}{x^2+xa+a^2}\times\dfrac{x^3-a^3}{x^3+a^3}$. Check.

2. $\dfrac{1-x^2}{1+a}\times\dfrac{1-a^2}{x+x^2}\times\left(1+\dfrac{x}{1-x}\right)$.

3. $\dfrac{x^2-9x+20}{x^2-6x}\times\dfrac{x^2-13x+42}{x^2-5x}$.

4. $\dfrac{4a^3-4ax^2}{3bc^2-3bx^2}\cdot\dfrac{bc+bx}{a^2-ax}$. Check.

5. $\dfrac{a^6-b^6}{(a-b)^2}\cdot\dfrac{a-b}{a^2+ab+b^2}$. Check.

6. $\left(\dfrac{1}{1+x}+\dfrac{x}{1-x}\right)\cdot\left(\dfrac{1}{1-x}-\dfrac{x}{1+x}\right)$.

7. $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right)\cdot\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

8. $\dfrac{a^2-b^2}{a^2-2ab+b^2}\cdot\dfrac{a-b}{a^2+ab}$. Check.

9. $\dfrac{a^2-6a}{a^2-9a+20}\cdot\dfrac{a^2-5a}{a^3-13a-138}$. Check.

10. $(ax+by+c)\cdot(\frac{1}{2}x+\frac{1}{3}y+\frac{1}{4})$. Check.

11. $\dfrac{x^2-5x+6}{x^2+3x-40}\cdot\dfrac{x^2+6x-16}{x^2-4x+3}\cdot\dfrac{x^2+4x-5}{x^2-7x+10}$.

12. $\left(x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}\right)\left(x-\dfrac{1}{x}\right)$. Check.

13. $\dfrac{x^{4n}+x^{2n}y^{2n}+y^{4n}}{x^{3n}-y^{3n}}\cdot\dfrac{x^n-y^n}{x^{2n}-x^ny^n+y^{2n}}$.

14. $\dfrac{x^3-3x+2}{x^3+2x^2+2x+1}\cdot\dfrac{x^2+2x+1}{x^3-5x+4}$.

15. $\dfrac{x^2-4x-21}{4x^2+12x+9}\cdot\dfrac{2x^2-5x-12}{x^2-4x-21}$.

16. $\dfrac{x^2-x-20}{x^2-x-2}\cdot\dfrac{2x^2+3x+1}{x^2+x-12}$.

17. $\dfrac{x^2-10x+21}{x^4-7x^2+1}\cdot\dfrac{x^3-8x-3}{x^3-46x-21}$.

18. $\dfrac{x^6-a^6}{x^2+(a+b)x+ab}\cdot\dfrac{x^2+(b+c)x+bc}{x^4+x^2a^2+a^4}$.

19. $\left(\dfrac{x}{z}-\dfrac{y}{s}+\dfrac{z}{t}\right)\left(\dfrac{x}{y}-\dfrac{z}{s}\right)$. Check.

20. $\left(\dfrac{4x^2}{a^2}+4+\dfrac{3a^3}{4x^2}\right)\cdot\left(\dfrac{2x}{a}+\dfrac{3a}{2x}\right)$.

21. $\left(\dfrac{a}{b}-\dfrac{b}{a}+\dfrac{c}{d}-\dfrac{d}{c}\right)\cdot\left(\dfrac{a}{b}-\dfrac{b}{a}-\dfrac{c}{d}+\dfrac{d}{c}\right)$.

22. $\left(\dfrac{x^2c}{y^2}-\dfrac{2x^3c}{y}-x^4c\right)\cdot\left(\dfrac{x}{y^3}+\dfrac{3x^2}{y^2}+c^2\right)$.

23. $\left(\dfrac{3xy}{2z}-\dfrac{5sm}{g}+8t\right)\cdot\left(\dfrac{3xy}{2z}+\dfrac{5sm}{g}-8t\right)$.

24. $\left(\frac{x^2}{2}-\frac{x}{3}+\frac{1}{4}\right)\left(\frac{2x}{3}-\frac{1}{2}\right)$.

25. $\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}-\frac{b}{a}-\frac{a}{b}+1\right)\cdot\left(\frac{b}{a}-\frac{a}{b}\right)$.

26. $\frac{x-(y-z)-\{x-y-z-2(z+y)\}}{2x^2+10xz+12z^2}\cdot\frac{x^2+xz-6z^2}{x-2z}$.

Section V

DIVISION

259. Problem 1. **To divide a fraction by an integral expression.**

Rule. *Divide the numerator, or multiply the denominator, by the integral expression.*

Dem. 1. Let $\frac{r}{s}$ represent the given fraction and a the integral expression.

Let $x =$ the quotient. Then

$$x=\frac{r\div a}{s},\ \text{or}\ \frac{r}{sa}.$$

For 1. $x=\frac{r}{s}\div a$, by assumption.

2. $sx=r\div a$, by clearing of fractions.

3. $x=\frac{r\div a}{s}$, by dividing both members of (2) by s.

4. Or, $x=\frac{r}{sa}$, by multiplying both terms of fraction in (3) by a.

Dem. 2. The numerator corresponds to the dividend, the denominator to the divisor, and the value of the fraction to the quotient. If the dividend be divided, or the divisor be multiplied, the quotient is divided. But the quotient is the value of the fraction. Hence the fraction is divided.

Model Solutions

1. Divide $\dfrac{a^{12}+b^{12}}{a^8+b^8}$ by a^4+b^4.

$$\frac{a^{12}+b^{12}}{a^8+b^8}\div(a^4+b^4)=\frac{a^8-a^4b^4+b^8}{a^8+b^8},$$ by dividing the numerator.

Questions. 1. Why are ( ) written about the divisor? 2. What would be the divisor if the ( ) were omitted? 3. What proposition in factoring may be applied to this example? 4. What are the laws by which such quotient is written? 5. What are the factors of a^8+b^8? 6. State the method of factoring it. 7. What are the factors of $a^8-a^4b^4+b^8$? 8. State the proposition by which it is factored. 9. In the quotient, why may not a^8 and b^8 in the numerator be canceled by the a^8 and b^8 in the denominator?

2. Divide $\dfrac{x+y}{4a^2-6a+9}$ by $(2a+3)$.

$$\frac{x+y}{4a^2-6a+9}\div(2a+3)=\frac{x+y}{8a^3+27},$$ by multiplying the denominator.

3. Divide $\dfrac{x^2-7x+6}{x^4-x^3+x^2-x+1}$ by (x^3-6x^2-x+6).

$$\frac{\cancel{(x-1)}\cancel{(x-6)}}{x^4-x^3+x^2-x+1}\div\cancel{(x-6)}(x+1)\cancel{(x-1)}=\frac{1}{x^5+1}.$$

Questions. 1. State proposition for factoring x^2-7x+6. 2. State propositions used in factoring x^3-6x^2-x+6. 3. Could any other be used? 4. How is the fraction divided by $x-6$? 5. By $x-1$? 6. By $x+1$? 7. How may it be known without actual multiplication that the product of $x^4-x^3+x^2-x+1$ and $x+1$ is x^5+1?

EXAMPLES

1. $\dfrac{x^6+a^6}{3}\div(x^2+a^2)$. Check.

2. $\dfrac{8a^3-1}{x+y}\div(2a-1)$. Check.

3. $\dfrac{x^2+x+1}{x^2-x+1}\div(x^4+x^2+1)$.

4. $\dfrac{(x^2-x+1)}{x+1}\div(x^3+1)$. Check.

5. $\dfrac{a^{5n}+b^{5n}}{a^n-b^n}\div(a^{2n}-b^{2n})$. Check.

6. $\dfrac{x^8+x^4y^4+y^8}{x^2-xy+y^2}\div(x^4-x^2y^2+y^4)$.

7. $\dfrac{a^{20}+b^{20}}{15} \div 3(a^{12}+b^{12})$. Check.

8. $\dfrac{4a^3-32a-12}{a^2-2a+1} \div (5a^3-40a-15)$.

9. $\dfrac{3x^3-2x^2-1}{6x+5} \div (6x^2-11x+5)$.

10. $\dfrac{x^4+x^3+x^2+x+1}{x^2+x+1} \div (x^5-1)$.

11. $\dfrac{x^{m+1}+x^my+xy^m+y^{m+1}}{x+y} \div (x^m+y^m)$.

12. $\dfrac{2x^2(y+z)^{2n}-\frac{1}{2}}{2x(y+z)+1} \div \{x(y+z)^n+\frac{1}{2}\}$.

13. $\dfrac{x^{12n}-y^{12n}}{x^{4n}-y^{4n}} \div (x^{4n}+x^{2n}y^{2n}+y^{4n})$. Check.

14. $\dfrac{5x^{3n}-15x^{2n}y^n+15x^ny^{2n}-5y^{3n}}{x^2-y^2} \div (x^n-y^n)(x^n+y^n)$.

15. $\dfrac{x^3+(y+z)x^2-xyz-yz(y+z)}{x^{2n}-y^{2n}} \div (x^2-yz)(x+y)$.

16. $\dfrac{x^8+2x^6y^2+3x^4y^4+2x^2y^6+y^8}{(x^2+xy+y^2)(x^6-y^6)} \div (x^2-xy+y^2)(x^2+xy+y^2)$.

260. PROBLEM 2. **To divide by a fraction.**

Rule 1. *Divide by the numerator and multiply the quotient by the denominator; or,*

Rule 2. *Multiply the divisor inverted by the dividend.*

Dem. 1. Let $\frac{r}{s}$ be the dividend, which may be either integral or fractional, and $\frac{a}{b}$ the fractional divisor.

Let $x=$ the quotient.

Then $$x=\frac{r\div a}{s\div b},\text{ or }\frac{br}{as}.$$

For 1. $x = \frac{r}{s} \div \frac{a}{b}$, by assumption.

2. $\frac{a}{b} \cdot x = \frac{r}{s}$, by multiplying both members of (1) by $\frac{a}{b}$.

3. $sax = br$, by clearing (2) of fractions.

4. $x = \frac{br}{as}$, by dividing both members of (3) by as.

5. Or, $x = \frac{r \div a}{s \div b}$, by dividing both terms of the fraction in (4) by ab.

Dem. 2. Let $\frac{r}{s} \div \frac{a}{b}$ represent a given problem. Then the quotient $= \frac{r \div a}{s \div b}$, or $\frac{br}{as}$.

For $\frac{r}{s} \div a = \frac{r \div a}{s}$, or $\frac{r}{as}$. Why?

But in dividing by a, a divisor has been used which is b times too large; $\therefore$ the quotient is b times too small. To make it right $\frac{r \div a}{s}$, or $\frac{r}{as}$, must be multiplied by b. This is done by dividing s by b, giving $\frac{r \div a}{s \div b}$, or by multiplying r by b, giving $\frac{br}{as}$. Why?

Dem. 3. Let $\frac{r}{s} \div \frac{a}{b}$ be a given problem. Then $\frac{br}{as}$ is the quotient.

For $1 \div \frac{a}{b} = \frac{b}{a}$, read "$\frac{a}{b}$ is contained in 1, $\frac{b}{a}$ times,"

and $\frac{r}{s} \div \frac{a}{b} = \frac{b}{a} \times \frac{r}{s}$, read "$\frac{r}{s}$ times $\frac{b}{a}$," $= \frac{br}{as}$.

For if $\frac{a}{b}$ is contained in 1, $\frac{b}{a}$ times, it will be contained in $\frac{r}{s}$, $\frac{r}{s}$ times as many times, or $\frac{br}{as}$ times.

Questions. 1. Why is the divisor inverted? *Ans.* To find how many times it is contained in 1. 2. Why then multiply by the dividend? *Ans.* If the divisor is contained in 1 a certain number of times, it will be contained in the dividend, the dividend times as many times. 3. If $\frac{r}{s} \div \frac{a}{b}$ is a given problem, is the first step taken in the operation strictly correct when written $\frac{r}{s} \times \frac{b}{a}$? Why should it be written $\frac{b}{a} \times \frac{r}{s}$? See Art. **15**.

261. Cor. 1. Mixed forms of expression can be reduced to fractional forms and the operation of division performed in the usual manner.

262. Cor. 2. One fractional polynomial may be divided by another in the same manner as polynomials in the integral form are divided.

263. Cor. 3. (1) When the numerator of a fraction becomes zero, the quotient is zero.

(2) When the denominator becomes zero, the operation of division is impossible, but the quotient is represented by the symbol ∞, called *infinity*.

(3) When both terms of the fraction become zero, the form is called an *indeterminate* one.

Dem. Let $x =$ the quotient of $\frac{m}{n}$.

(1) When $m = 0$, $x = \frac{0}{n} = 0$. $(n \neq 0.)$

For $\quad nx = 0$, by clearing of fractions

$\therefore\ x = 0$; $\because$ if product $= 0$, x must $= 0$, if $n \neq 0$.

(2) When $n = 0$, $x = \frac{m}{0} = \infty$. $(m \neq 0.)$

For $\frac{m}{1} = m,\quad \frac{m}{\frac{1}{2}} = 2m,\quad \frac{m}{.01} = 100m,$

$$\frac{m}{.000000001} = 1000000000m, \text{ etc.,}$$

till the denominator becomes practically zero, while the quotient becomes infinite, and may be represented by ∞.

(3) When $m = 0$ and $n = 0$, $x = \frac{0}{0} = 0$. (Any quantity.)

For, if $x = \frac{0}{0}$, then $0 \cdot x = 0$. Why?

Now any value of x will satisfy the equation.

$\therefore\ x = \frac{0}{0} = a$, in which a stands for any quantity.

Model Solutions

1. $(a^2 - b^2 - 2bc - c^2) \div \frac{a^2 - b^2 - ac - bc}{a + b + c}$

$$= \frac{(a+b+c)\cancel{(a-b-c)}}{1} \times \frac{(a+b+c)}{(a+b)\cancel{(a-b-c)}} \quad \text{Why?}$$

$$= \frac{(a+b+c)^2}{a+b}. \quad \text{Why?}$$

2. $\frac{x^{3n} - y^{3n}}{x^{3n} + y^{3n}} \div \frac{x^n - y^n}{x^n + y^n}$

$$= \frac{x^{2n} + x^n y^n + y^{2n}}{x^{2n} - x^n y^n + y^{2n}},$$ by dividing the 1st numerator by the 2d, and the 1st denominator by the 2d.

3. $\frac{x-5}{2x+3} \div \frac{3x-7}{x+3}.$

$$\frac{x-5}{2x+3} \div (3x-7) = \frac{x-5}{6x^2 - 5x - 21}. \quad \text{Why?}$$

$$\frac{x-5}{6x^2 - 5x - 21} \cdot (x+3) = \frac{x^2 - 2x - 15}{6x^2 - 5x - 21}. \quad \text{Why?}$$

4. $\dfrac{2x^2+7x-15}{6x^2+19x+15} \div \dfrac{2x^2+x-6}{3x^2-16x-35} \div \dfrac{x^2-2x-35}{2x^2+7x+6}$

$$= \frac{\cancel{(2x-3)}\cancel{(x+5)}}{\cancel{(3x+5)}\cancel{(2x+3)}} \cdot \frac{\cancel{(3x+5)}\cancel{(x-7)}}{\cancel{(2x-3)}\cancel{(x+2)}} \cdot \frac{\cancel{(2x+3)}\cancel{(x+2)}}{\cancel{(x+5)}\cancel{(x-7)}}$$

$= 1$, by cancellation.

5. $2 \div \left[\dfrac{a^3-2a^2b+ab^2}{a^3-ab^2} - 1\right]$

$$= 2 \div \left[\frac{a^3-2a^2b+ab^2}{a^3-ab^2} - \frac{a^3-ab^2}{a^3-ab^2}\right]$$

$$= 2 \div \left[\frac{-2a^2b+2ab^2}{a^3-ab^2}\right]$$

$$= 2 \div \left[\frac{-2\cancel{a}b\cancel{(a-b)}}{\cancel{a}(a+b)\cancel{(a-b)}}\right]$$

$$= 2 \div \frac{-2b}{a+b} \qquad \text{Why?}$$

$$= \frac{a+b}{-\cancel{2}b} \times \frac{\cancel{2}}{1} \qquad \text{Why?}$$

$$= \frac{a+b}{-b} \qquad \text{Why?}$$

$$= -\frac{a+b}{b} \qquad \text{Why?}$$

6. $\left(\dfrac{a^3}{3} - \dfrac{17a^2}{36} + \dfrac{a}{3} - \dfrac{1}{8}\right) \div \left(\dfrac{2a}{3} - \dfrac{1}{2}\right)$.

$$\begin{array}{l|l}
\dfrac{a^3}{3} - \dfrac{17a^2}{36} + \dfrac{a}{3} - \dfrac{1}{8} & \dfrac{2a}{3} - \dfrac{1}{2} \\ \cline{2-2}
\dfrac{a^3}{3} - \dfrac{a^2}{4} & \dfrac{a^2}{2} - \dfrac{a}{3} + \dfrac{1}{4} = \text{Quotient.} \\ \cline{1-1}
\qquad -\dfrac{2a^2}{9} + \dfrac{a}{3} & \\
\qquad -\dfrac{2a^2}{9} + \dfrac{a}{6} & \\ \cline{1-1}
\qquad\qquad\quad +\dfrac{a}{6} - \dfrac{1}{8} & \\
\qquad\qquad\quad +\dfrac{a}{6} - \dfrac{1}{8} & \\ \cline{1-1}
\end{array}$$

7. Ex. 6 by synthetic division. $\frac{2a}{3}-\frac{1}{2}\equiv\frac{2}{3}\left(a-\frac{3}{4}\right).$

$$\begin{array}{l|l}
\frac{1}{3}-\frac{17}{36}+\frac{1}{3}-\frac{1}{8} & -\frac{3}{4} \\
\quad -\frac{1}{4}+\frac{1}{6}-\frac{1}{8} & \\
\hline
\left(\frac{1}{3}-\frac{2}{9}+\frac{1}{6}\right)\div\frac{2}{3}=\frac{a^2}{2}-\frac{a}{3}+\frac{1}{4}= &
\end{array}$$

Quotient. Why?

8. $\left(1+\frac{y}{x+y}+\frac{x}{y}\right)\div\left(2+\frac{x}{y}-\frac{x}{x+y}\right)$

$$=\frac{(xy+y^2)+y^2+(x^2+xy)}{(x+y)y}\div\frac{(2xy+2y^2)+(x^2+xy)-xy}{(x+y)y}$$

$$=\frac{x^2+2xy+2y^2}{(x+y)y}\div\frac{x^2+2xy+2y^2}{(x+y)y}=1.$$ Or

$\left(1+\frac{y}{x+y}+\frac{x}{y}\right)\div\left(2+\frac{x}{y}-1+\frac{y}{x+y}\right)=1.$ State reasons.

9. Find the value of $\frac{x^3-a^3}{x^2-a^2}$ when $x=2,\ a=2.$

By direct substitution, $\frac{x^3-a^3}{x^2-a^2}\equiv\frac{8-8}{4-4}\equiv\frac{0}{0}$, an indeterminate form.

But $$\frac{x^3-a^3}{x^2-a^2}\equiv\frac{(x-a)(x^2+xa+a^2)}{(x-a)(x+a)}\equiv\frac{x-a}{x-a}\cdot\frac{x^2+xa+a^2}{x+a}$$

$$\equiv\frac{x^2+xa+a^2}{x+a}\equiv\frac{4+4+4}{2+2}\equiv\frac{12}{4}\equiv 3.$$

EXAMPLES

1. $\left(\frac{x}{1+x}+\frac{1-x}{x}\right)\div\left(\frac{x}{1+x}-\frac{1-x}{x}\right).$

2. $\left(\frac{c-b}{c+b}-\frac{c^3-b^3}{c^3-b^3}\right)\div\left(\frac{c+b}{c-b}+\frac{c^2+b^2}{c^2-b^2}\right).$

3. $\left(\frac{b^2}{a}-8a+\frac{12a^3}{b^2}\right)\div\left(\frac{b}{a}-\frac{6a}{b}\right).$

4. $\frac{a^4-x^4}{a^2-2ax+x^2}\div\frac{a^2x+x^3}{a^3-x^3}.$

5. $\frac{xy^2+y^3}{x^2+xy+y^2}\div\frac{x^2y^2-x^4}{y^3-x^3}.$

6. $\frac{x-2}{x^2-x}\cdot\frac{x^3+x^2y}{xy-y^2}\div\frac{xy^2+y^3}{x^3-x^2y}.$

7. $\left(x^5-\frac{1}{x^5}\right)\div\left(x-\frac{1}{x}\right)$.

8. $\left(\frac{a^2}{b^3}-\frac{1}{a}\right)\div\left(\frac{a}{b^2}+\frac{1}{b}+\frac{1}{a}\right)$.

9. $\frac{4\,a^6b-4\,b^7}{a^2b-2\,ab^2+b^3}\div\frac{a^2+ab+b^2}{a-b}$.

10. $\left(\frac{4\,a^2}{b^2}+4+\frac{3\,b^2}{4\,a^2}\right)\div\left(\frac{2\,a}{b}+\frac{3\,b}{2\,a}\right)$.

11. $\left(\frac{1-x^3}{1+x^3}-\frac{1-x}{1+x}\right)\div\left(\frac{1+x^2}{1-x^2}+\frac{1+x}{1-x}\right)$.

12. $\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$.

13. $\left\{1+\left(\frac{a-b}{a+b}\right)^2\right\}\div\left\{1-\left(\frac{a-b}{a+b}\right)^2\right\}$.

14. $\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}-\frac{c^2}{d^2}-\frac{d^2}{c^2}\right)\div\left(-\frac{a}{b}+\frac{b}{a}+\frac{c}{d}-\frac{d}{c}\right)$.

15. $\left(\frac{1}{x^3}-\frac{3}{x^2y}+\frac{3}{xy^2}-\frac{1}{y^3}\right)\div\left(\frac{1}{x}-\frac{1}{y}\right)$.

16. $\left(\frac{1}{x^4}+\frac{1}{x^2y^2}+\frac{1}{y^4}\right)\div\left(\frac{1}{x^2}-\frac{1}{xy}+\frac{1}{y^2}\right)$.

17. $\left(\frac{a^5}{x^5}+\frac{5\,a^4}{x^4}+\frac{10\,a^3}{x^3}+\frac{10\,a^2}{x^2}+\frac{5\,a}{x}+1\right)\div\left(\frac{a}{x}+1\right)$.

18. $\left(\frac{a}{a-b}-\frac{b}{a+b}\right)\div\left(\frac{b}{a-b}-\frac{a}{a+b}\right)$.

19. $\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{2}{xy}+\frac{2}{xz}+\frac{2}{yz}\right)\div\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$.

20. $\left\{\frac{x+y}{x-y}+\frac{x^2+y^2}{x^2-y^2}\right\}\div\left\{\frac{x-y}{x+y}-\frac{x^3-y^3}{x^3+y^3}\right\}$.

21. $\left\{m^2-\frac{(m^2+n^2-p^2)^2}{4\,n^2}\right\}\div\left\{\left(\frac{m+p}{n}\right)^2-1\right\}$

22. $\dfrac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2(x^3+x^2y+xy^2)}+\dfrac{x+y}{2xy(x^3-y^3)}.$

23. $\dfrac{c^2-6cd+9d^2}{c^2-4cd+4d^2}\div\left[\dfrac{c^2-9d^2}{c^2-4d^2}\div\dfrac{c^2+cd-6d^2}{c^2-cd-6d^2}\right].$

Section VI

SIMPLIFICATION OF COMPLEX FRACTIONS

264. Problem 1. **To reduce a complex fraction to a simple one.**

Rule 1. *Multiply both terms of the complex fraction by the L. C. M. of the denominators of the partial fractions; or*

Rule 2. *Perform the operations indicated in the complex fraction.*

Dem. Multiplying both terms of a fraction by the same number does not change the value of the fraction.

Model Solutions

1. Simplify $\dfrac{3\frac{2}{5}-\frac{5}{6}(m-2)}{\frac{3}{10}+\frac{4}{15}(m+2)}.$

$$\frac{\frac{17}{5}-\frac{5}{6}(m-2)\cdot 30}{\frac{3}{10}+\frac{4}{15}(m+2)\cdot 30}=\frac{102-25(m-2)}{9+8(m+2)}=\frac{152-25m}{25+8m}.$$

2. Simplify $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}.$

$$\frac{1}{x+\dfrac{1\cdot(3-x)}{\left(1+\dfrac{x+1}{3-x}\right)\cdot(3-x)}}=\frac{1}{x+\dfrac{3-x}{3-x+x+1}}=\frac{1\cdot 4}{\left(x+\dfrac{3-x}{4}\right)\cdot 4}$$

$$=\frac{4}{4x+3-x}=\frac{4}{3x+3}.$$

3. Simplify $\dfrac{\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}}{\dfrac{a^2-b^2}{a^2+b^2}+\dfrac{a^2+b^2}{a^2-b^2}}$.

$$\frac{\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}}{\dfrac{a^2-b^2}{a^2+b^2}+\dfrac{a^2+b^2}{a^2-b^2}}=\frac{\dfrac{2a^2+2b^2}{a^2-b^2}}{\dfrac{2a^4+2b^4}{a^4-b^4}}=\frac{a^4-b^4}{2(a^4+b^4)}\cdot\frac{2(a^2+b^2)}{a^2-b^2}=\frac{(a^2+b^2)^2}{a^4+b^4}.$$

265. PROBLEM 2. **To reduce a continued fraction to a simple one.**

Rule. *Begin with the lowest complex form and simplify step by step, as in Problem* 1.

Dem. This is a direct consequence of Problem 1, because each step involves the simplification of a complex fraction.

MODEL SOLUTION

Reduce $\dfrac{\dfrac{x^2}{1+\dfrac{1}{x-1}}}{1+\dfrac{x}{x^2-\dfrac{1}{1-\dfrac{x-1}{x}}}}$ to its simplest form.

$$\frac{x^2}{\left(1+\dfrac{1}{x-1}\right)}\frac{\cdot(x-1)}{\cdot(x-1)}=\frac{x^2(x-1)}{x-1+1}=\frac{x^2(x-1)}{x}=x(x-1),$$ new numerator.

$$\frac{1}{\left(1-\dfrac{x-1}{x}\right)}\frac{\cdot x}{\cdot x}=\frac{x}{x-x+1}=x,$$ lowest complex form reduced.

$$1+\frac{x}{x^2-x}=1+\frac{1}{x-1},$$ new denominator.

$$\frac{x(x-1)}{\left(1+\dfrac{1}{x-1}\right)}\frac{\cdot(x-1)}{\cdot(x-1)}=\frac{x(x-1)^2}{x-1+1}=(x-1)^2,$$ simplest form.

EXAMPLES

Reduce to the simplest form:

1. $\dfrac{3\frac{1}{2}-\frac{3}{4}(x+3)}{\frac{2}{3}+\frac{3}{5}(x-2)}$. Check.

2. $\dfrac{\dfrac{2x}{x^2+1}+\dfrac{2x}{x^2-1}}{\dfrac{x}{x^2+1}-\dfrac{x}{x^2-1}}$. Check.

3. $\dfrac{\dfrac{x^3+y^3}{x^2-y^2}}{\dfrac{x^2-xy+y^2}{x-y}}$. Check.

4. $\dfrac{a+\dfrac{x-a}{1+ax}}{1-\dfrac{a(x-a)}{1+ax}}$. Check.

5. $\dfrac{\dfrac{2x+y}{x+y}-1}{1-\dfrac{x}{x+y}}$. Check.

6. $\dfrac{1\frac{1}{2}-2\frac{1}{3}(3-x)}{\frac{2}{3}(x-3)-\frac{3}{4}}$. Check.

7. $\dfrac{\dfrac{1}{1+x}+\dfrac{1}{1-x}}{\dfrac{1}{1+x}-\dfrac{1}{1-x}}$. Check.

8. $\dfrac{a^2+a+1}{\dfrac{1}{a^2}+\dfrac{1}{a}+1}$. Check.

9. $\dfrac{\dfrac{1}{x^4}+\dfrac{1}{x^2y^2}+\dfrac{1}{y^4}}{\dfrac{1}{x^2}+\dfrac{1}{xy}+\dfrac{1}{y^2}}$. Check.

10. $\dfrac{x^6+\dfrac{1}{x^6}}{x^2+\dfrac{1}{x^2}}$. Check.

11. $\dfrac{4x^2+14x+\dfrac{98x-27}{2x-7}}{\dfrac{1}{6}+\dfrac{6x-9\frac{1}{2}}{12x^2-18x-84}}$.

12. $\dfrac{\dfrac{8x^3-27}{6x^2+5x-6}\cdot\dfrac{x^{10}+2^{10}}{2x-3}}{\dfrac{4x^2+6x+9}{3x-2}\div\dfrac{2x+3}{x^2+4}}$.

13. $\dfrac{3a\left\{\dfrac{\overline{x-y}\;\overline{x+y}}{x^2-y^2}-\dfrac{a^2-ab+a}{a}\right\}-\dfrac{12a^3-12a^2b}{-4a}}{6a-\dfrac{4b}{7c-\frac{3}{4}}+5a+16b+\dfrac{28}{7c}}$.

14. $$\frac{\dfrac{x^6-a^6}{x^6+a^6}\div\dfrac{x^4+x^2a^2+a^4}{x^2+a^2}}{\dfrac{x+a}{x^2+ax\sqrt{3}+a^2}\cdot\dfrac{x-a}{x^2-ax\sqrt{3}+a^2}}.$$

15. $$\frac{1}{x+y}\div\left\{\frac{y}{2}\left(\frac{1}{x-y}+\frac{1}{x+y}\right)\cdot\frac{x^2-y^2}{x^2y-y^2x}\right\}.$$

16. $$\frac{x^2\left(\dfrac{1}{z^2}-\dfrac{1}{y^2}\right)+y^2\left(\dfrac{1}{x^2}-\dfrac{1}{z^2}\right)+z^2\left(\dfrac{1}{y^2}-\dfrac{1}{x^2}\right)}{\dfrac{1}{yz}\left(\dfrac{1}{z}-\dfrac{1}{y}\right)+\dfrac{1}{zx}\left(\dfrac{1}{x}-\dfrac{1}{z}\right)+\dfrac{1}{xy}\left(\dfrac{1}{y}-\dfrac{1}{x}\right)}.$$

17. $$\frac{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{ab}}}}{ab-\dfrac{1+a^2b^2}{ab}}.$$

18. $$\frac{1}{x+\dfrac{2}{1-\dfrac{x-2}{2x+1}}}.$$

19. $$\frac{\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}}{\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}}.$$

20. $$\frac{1-\dfrac{2}{3-\dfrac{4}{5-\frac{6}{7}}}}{a-\dfrac{b}{c-\dfrac{d}{e-\dfrac{f}{g}}}}.$$

21. $$\left\{\frac{y+\dfrac{x-y}{1+xy}}{1-\dfrac{y(x-y)}{1+xy}}-\frac{x-\dfrac{x-y}{1-xy}}{1-\dfrac{x(x-y)}{1-xy}}\right\}\div\left\{\frac{x}{y}-\frac{y}{x}\right\}.$$

22. $$\frac{\dfrac{2x+y^3}{2x-y^3}+\dfrac{4x^2+y^6}{4x^2-y^6}}{\dfrac{2x-y^3}{2x+y^3}-\dfrac{8x^3-y^9}{8x^3+y^9}}.$$

24. $$\frac{\dfrac{a^2}{b^3}+\dfrac{1}{a}}{\dfrac{a}{b}-\dfrac{1}{b}+\dfrac{1}{a}}.$$ Check.

23. $$\frac{1}{12+\dfrac{1}{1+\dfrac{1}{2+\frac{1}{3}}}}.$$

25. $$\frac{\dfrac{1}{x}-\dfrac{x+a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a+x}{a^2+x^2}}+\frac{\dfrac{1}{x}-\dfrac{x-a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a-x}{a^2+x^2}}.$$

26. $$\frac{\dfrac{m-[m-(n+m)-\{m-(m-\overline{n-m})\}]}{n(a+b)+am+bm}}{\dfrac{(a+b)(n-m)}{n^2-m^2}}.$$

MISCELLANEOUS EXAMPLES

Perform the operations indicated and simplify:

1. $\dfrac{2}{\frac{1}{x}-\frac{1}{y}}+\dfrac{y}{1-\frac{y}{x}}-\dfrac{x}{1-\frac{x}{y}}.$

2. $\dfrac{1}{a-b}+\dfrac{1}{a+b}-\dfrac{2a}{b^2-a^2}.$

3. $\dfrac{1-\dfrac{2xy}{(x+y)^2}}{1+\dfrac{2xy}{(x-y)^2}}\div\left\{\dfrac{1-\frac{y}{x}}{1+\frac{y}{x}}\right\}^2.$

4. $\dfrac{3x^2+12x+9}{x^4+4x^3-12x-9}.$

5. $\dfrac{m^4-2mn^3-m^3n+2n^4}{m^2-n^2}.$

6. $\dfrac{\dfrac{x}{1+x}+\dfrac{1-x}{x}}{\dfrac{x}{1+x}-\dfrac{1-x}{x}}.$ Check.

7. $\dfrac{16x^4-53x^2+45x+6}{8x^4-30x^3+31x^2-12}.$ Check.

8. $\dfrac{a+\dfrac{b}{1+\frac{a}{b}}}{a-\dfrac{b}{1-\frac{a}{b}}}\cdot(a^6-b^6).$ Check.

9. $\dfrac{\frac{2}{3}a^2-\frac{1}{4}a^2}{\frac{5}{6}a^2y^2+\frac{10}{24}a^2x^2}.$ Check.

10. $\left(\dfrac{1}{a}-\dfrac{1}{b}+\dfrac{1}{c}\right)\cdot\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right).$

11. $\dfrac{5}{2x+2}-\dfrac{1}{10x-10}-\dfrac{24}{10x+15}.$

12. $\dfrac{x^3-y^3}{x^4-y^4}-\dfrac{x-y}{x^2-y^2}-\dfrac{1}{2}\left(\dfrac{x+y}{x^2+y^2}-\dfrac{1}{x+y}\right).$

13. $\dfrac{1}{x+\dfrac{1}{y+\frac{1}{z}}}\div\dfrac{1}{x+\frac{1}{y}}-\dfrac{1}{y(xyz+x+z)}.$

14. $(1+\frac{1}{2}x+\frac{1}{4}x^2)\cdot(1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3).$

15. $\dfrac{2}{a-b}+\dfrac{2}{b-c}+\dfrac{2}{c-a}+\dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}.$

16. $$\frac{\dfrac{3}{abc}}{\dfrac{1}{bc}+\dfrac{1}{ca}-\dfrac{1}{ab}}-\frac{3-a-b+c}{a+b-c}.$$

17. $$\left\{\frac{x+2y}{x+y}+\frac{x}{y}\right\}\div\left\{\frac{x+2y}{y}-\frac{x}{x+y}\right\}.$$

18. $$\frac{2x^2+5x+2}{x^2-4}\div\frac{2x^2+9x+4}{x+4}.$$

19. $$\left(\frac{1}{a^3}+\frac{1}{b^3}-\frac{1}{c^3}+\frac{3}{abc}\right)\div\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right).$$

20. $$\frac{\dfrac{x}{y}+\dfrac{y}{x}-1}{\dfrac{x^3}{y^3}-1}\cdot\frac{\dfrac{x}{y}+\dfrac{y}{x}+1}{\dfrac{x^3}{y^3}+1}\cdot\left(\frac{x^4}{y^4}-\frac{x^2}{y^2}\right).$$

21. $$\left\{\frac{x^2}{x^2-y^2}-\frac{y^2}{x^2+y^2}\right\}\div\left\{\frac{(x^2-y^2)^2}{(x^2-y^2)^2+(x^2+y^2)^2}\right\}.$$

22. $$\frac{1}{x+3}+\frac{x-1}{x^2-3x+9}+\frac{x^2+x+1}{x^3+27}.$$

23. $$\frac{1}{12+\dfrac{1}{1+\dfrac{1}{2+\frac{1}{3}}}}+\frac{1}{2+\dfrac{1}{1+\dfrac{1}{4+\frac{1}{5}}}}.$$

24. $$\frac{\dfrac{1-x^3}{1+x^3}-\dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2}+\dfrac{1+x}{1-x}}.$$ Check.

25. $$\frac{6x^5-2x^4-11x^3+5x^2-10x}{9x^5+3x^4-11x^3+9x^2-10x}.$$

26. $$\frac{1-\frac{1}{2}(1-\frac{1}{3}(1-x))}{1-\frac{1}{3}(1-\frac{1}{2}(1-x))}.$$

27. $$\frac{\dfrac{1}{a}}{\dfrac{1}{a}-\dfrac{1}{b}}\div\frac{(a+b)^2-4ab}{\dfrac{1}{a^2}+\dfrac{1}{b^2}-\dfrac{2}{ab}}.$$

28. $$\frac{x-3a+\dfrac{4a^2}{a+x}}{x-\dfrac{2a^2}{a+x}}.$$

29. $$\frac{6x^4-13x^3+3x^2+2x}{6x^4+9x^3-30x^2-9x}.$$

30. $$\frac{1}{(1-x)(1-x^2)^2}\div\left\{\frac{1}{(1-x)^2}-\frac{1}{(1-x)(1-x^2)}\right\}.$$

31. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ca}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}.$

32. $\dfrac{1}{x(x-a)(x-b)}+\dfrac{1}{a(a-x)(a-b)}+\dfrac{1}{b(b-x)(b-a)}.$

33. $\dfrac{x^4+2x^2+9}{x^4-4x^3+10x^2-12x+9}.$

34. $\left\{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}\right\}\cdot\dfrac{x+y}{2y}.$

35. $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}.$

36. $\left\{a+\dfrac{b-a}{1+ab}\right\}\dfrac{a}{b}\div\left\{1-a\cdot\dfrac{b-a}{1+ab}\right\}.$

37. $\dfrac{x-4+\dfrac{6}{x+1}}{x-\dfrac{5}{x-1}}\cdot\dfrac{1-\dfrac{x+5}{x^2-1}}{\dfrac{(x-1)(x-2)}{x+1}}.$

38. $\dfrac{\dfrac{x^2+y^2+2xy-z^2}{4x^2y^2-(x^2+y^2-z^2)^2}}{\dfrac{1}{x^2+y^2+z^2+2xz-2xy-2yz}}.$

39. $\dfrac{\dfrac{c-b}{b+c}-\dfrac{c^2-b^2}{b^2+c^2}}{\dfrac{b+c}{c-b}+\dfrac{c^2+b^2}{c^2-b^2}}.$

40. $\dfrac{\dfrac{a}{c}+\dfrac{b}{d}}{\dfrac{a^2}{c^2}-\dfrac{b^2}{d^2}}\div\dfrac{cd}{e+f}.$

41. $\dfrac{x^{2n}-a^{2n}}{x^{3n}+a^{3n}}.$ Check.

42. $\dfrac{1-\dfrac{(a^2+b^2-c^2-d^2)^2}{4(ab+cd)^2}}{\dfrac{(a+b)^2-(c-d)^2}{4(ab+cd)}}.$

43. $\dfrac{x^{2n}-y^{2n}}{x^{3n}+y^{3n}}\div\dfrac{x^n-y^n}{x^n+y^n}.$

44. $\dfrac{\dfrac{x^4-y^4}{x^2-2xy+y^2}}{\dfrac{x^2+xy}{x-y}}.$

45. $\left(\dfrac{x^2}{y^2}-1\right)\left(\dfrac{x}{x-1}\right)\left(\dfrac{x^3}{y^3}-1\right)\left(\dfrac{x^2+xy}{x^2+xy+y^2}-1\right).$

46. $\left\{x-\dfrac{y(x-y)}{x+y}\right\}\div\left\{1-\dfrac{xy-y^2}{x^2}\right\}\cdot\left\{x-\dfrac{y^2(x-y)}{x^2+y^2}\right\}.$

EXAMPLES IN SUBSTITUTION

Find the value of :

1. $x^3 - a^3$ when $x = 1 - b$, and $a = 1 + b$.
2. $a^2 - 2ax + x^2$ when $a = \frac{2}{3}$, and $x = \frac{5}{2}$.
3. $x^3 + 3x^2y + 3xy^2 + y^3$ when $x = a - 1$, and $y = a + 1$.
4. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ when $x = 2c - 1$, $y = 2c - 2$, $z = 2c - 3$.
5. $\dfrac{x-a}{x+b}$ when $x = \dfrac{a}{b}$. Check.
6. $\dfrac{x + px - qx}{p - q} - \dfrac{mx - n}{m}$ when $x = \dfrac{n(q-p)}{m}$.
7. $bx + 2x - a - 3x - 2c$ when $x = \dfrac{a - 2c}{b - 1}$.
8. $\dfrac{x^4 - 1}{x^2 - 1}$, $\dfrac{x^2 - 1}{x^3 - 1}$, and $\dfrac{x^3 - 1}{x + 1}$ when $x = 1$.
9. $ax + b - \dfrac{x}{a} + \dfrac{1}{a}$ when $x = \dfrac{a(1 - b^2)}{b(a^2 - 1)}$.
10. $\dfrac{1}{a} + \dfrac{b}{x} - c$ when $x = \dfrac{ab}{ac - 1}$.
11. $\dfrac{x^3 + x^2 - 2}{x^3 - x^2}$ when $x = 0$. Check.
12. $\dfrac{ab}{x} - bc - \dfrac{1}{x}$ when $x = \dfrac{ab - 1}{bc + m}$.
13. $\dfrac{x^2 + xa - 2a^2}{x^2 - a^2}$ when $x = a$.
14. $\dfrac{x}{a} + 3b - \dfrac{dx}{c}$ when $x = \dfrac{ac(1 - 3b)}{c - ad}$.
15. $\dfrac{1}{2}\left(x - \dfrac{a}{3}\right) - \dfrac{1}{3}\left(x - \dfrac{a}{4}\right) + \dfrac{1}{4}\left(x - \dfrac{a}{5}\right)$ when $x = \dfrac{8a}{25}$.
16. $\dfrac{x-a}{3} - \dfrac{2x - 3b}{5} - \dfrac{a - x}{2} - 10a - 11b$ when $x = 25a + 24b$.

SYNOPSIS FOR REVIEW, CHAPTER VI

FRACTIONAL EXPRESSIONS

- **SECT. I Classification and Definitions**
  - Fraction, **230**; Terms, **231**; Value, **232**; Reciprocal, **233**.
  - Fractions: Common, **234**; Decimal, **235**; Simple, **236**; Compound, **237**; Complex, **238**; Continued, **239**; Proper, **240**; Improper, **241**.
  - Sign of a fraction, **242**.
- **SECT. II Reduction**
  - Reduction, **243**. Lowest Terms, **244**.
  - Prins.
    - Dividing or multiplying both terms by the same number, **246**. Dems.
    - Change of signs in a fraction, **247**. Dem.
  - Cyclic Order of denominators, **247**.
  - Probs.
    1. To reduce to lowest terms, **245**. Rules. Dems.
    2. To reduce improper fractions to whole or mixed numbers, **248**. Rule. Dem.
    3. To reduce fractions to L. C. D., **249**. Rule. Dem.
- **SECT. III Addition and Subtraction**
  - Prob. To add and subtract fractions, **250**. Rule. Dem.
  - Five suggestions to aid in addition and subtraction, **251**.
  - Cors.
    1. Integral and mixed forms reduced to fractional forms, **252**. Dem.
    2. Mixed forms added and subtracted, **253**.
- **SECT. IV Multiplication**
  - Probs.
    1. To multiply a fraction by an integer, **254**. Rule. Dems.
    2. To multiply by a fraction, **255**. Rules. Dems.
  - Cors.
    1. Any number of fractions multiplied together, **256**. Dem.
    2. Mixed forms multiplied together, **257**.
    3. Fractional polynomials multiplied together, **258**.
- **SECT. V Division**
  - Probs.
    1. To divide a fraction by an integer, **259**. Rule. Dems.
    2. To divide by a fraction, **260**. Rules. Dems.
  - Cors.
    1. Mixed forms of expression divided, **261**.
    2. Fractional polynomials divided, **262**.
    3. Forms of expression like $\frac{0}{a}$, $\frac{a}{0}$, $\frac{0}{0}$, **263**. Dem.
- **SECT. VI Simplification of Complex Fractions**
  - Probs.
    1. To reduce a complex fraction to a simple one, **264**. Rules. Dem.
    2. To reduce a continued fraction to a simple one, **265**. Rule. Dem.

Sample Test Questions

1. Define fraction: common, decimal, simple, complex, proper, improper.
2. Define reciprocal, numerator, denominator, value of a fraction.
3. Define reduction, mixed forms, lowest terms, L. C. D., continued fraction.
4. Explain the meaning of $b \times \frac{a}{b} = a$; of $\frac{0}{a}$; $\frac{a}{0}$; $\frac{0}{0}$; of cyclic order.
5. Give the diagram of fractions; the principle for multiplying both terms.
6. How is a fraction reduced to its lowest terms? State the reasons.
7. How are fractions reduced to the L. C. D.? Why is the L. C. D. divided by each denominator?
8. How are fractions added? State the reasons for each step.
9. How are fractions subtracted? State the reasons for each step.
10. How is a fraction multiplied by an integral expression? Why?
11. How is a fraction multiplied by a fraction? Why?
12. How is an integral expression multiplied by a fraction? Why?
13. How is an integral expression divided by a fraction? Why?
14. How is a fraction divided by an integral expression? Why?
15. How is a fraction divided by a fraction? Why?
16. Why are the terms of the divisor inverted? What is the next step? Why?
17. How are mixed forms reduced to fractional forms? Why?
18. How are improper fractions reduced to whole or mixed forms? Why?
19. How are fractional polynomials multiplied? State the laws involved.
20. How are fractional polynomials divided? State the laws involved.
21. How are mixed forms added? How subtracted?
22. How are complex fractions reduced to simple ones? Why?
23. How are continued fractions reduced to simple ones? Why?
24. $+\frac{+}{+} = ?$ $\quad -\frac{-}{-} = ?$ $\quad +\frac{-}{+} = ?$ $\quad -\frac{-}{+} = ?$ $\quad -\frac{+}{-} = ?$
25. How are mixed forms divided? State the reasons for the process.
26. How are mixed forms multiplied? State the reasons for the process.
27. Does the definition "One or more of the equal parts into which a unit is divided" apply to $\frac{x}{y}$? Why?
28. What part of the subject of fractions do you understand the least? Why does it cause you trouble?
29. State and prove the effect of multiplying the numerator by an integer.
30. State and prove the effect of multiplying the denominator by an integer.
31. State and prove the effect of dividing the denominator by an integer.
32. State and prove the effect of dividing the numerator by an integer.
33. State and prove the effect of dividing both terms by an integer.
34. State and prove the effect of multiplying both terms by an integer.

CHAPTER VII

INEQUALITIES, AND MAXIMA AND MINIMA

INEQUALITIES

266. One number a is greater than another number b when their algebraic difference $a - b$ is positive.

ILLUSTRATIONS.

1. $5 > 3$ if $5 - 3$ is positive.
2. $-7 > -9$ if $-7 - (-9)$ is positive. Is it positive?
3. $0 > -4$ if $0 - (-4)$ is positive. Is it positive?
4. $0 < 3$ if $0 - 3$ is negative. Is it negative? Why?
5. $-6 < 0$ if $-6 - 0$ is negative. Is it negative?

Questions. 1. What are the symbols of relation? 2. How is each one read? 3. Describe each one. 4. Read $a > b < c_1 = d' \neq e_2 \not< f^2 \not> g'' :: h : i \equiv 10$. 5. Have such expressions as \$6 > \$2, \$5 ≡ \$5, $4 = 4$ any value? Why? Is it proper to speak of the *value* of an equation?

267. An **Inequality** is an expression in mathematical symbols, of inequality between two numbers or sets of numbers.

ILLUSTRATIONS. 1. $7 < 10$. 2. $5 > 3 > 2 > 1$.

268. The **First Member** of an inequality is the expression at the left of the inequality sign.

269. The **Second Member** of an inequality is the expression at the right of the inequality sign.

270. Inequalities are said to be in the **same sense** when they have the same symbol of inequality.

ILLUSTRATIONS. 1. $2>1$, $7>5>3$, $a>b>c$.

2. $2<3$, $3<5$, $a<b<c$.

271. Inequalities are said to be in the **opposite sense** when they have opposite symbols of inequality.

ILLUSTRATIONS. 1. $2<10$, $12>5$. 2. $a>b$, $c<d$.

272. Prop. 1. The sense of an inequality is not changed—

1. By adding equals to both members, or subtracting equals from both.

2. By multiplying or dividing both members by equal positive numbers.

3. By adding or multiplying together the corresponding members of two inequalities which are in the same sense, when all the members are positive.

4. By raising both members to the same power, when both are positive.

5. By raising both members to the same odd power in all cases.

6. By extracting the same odd root of both members.

7. By extracting the same even root of both members, if only the positive roots are compared.

Demonstration.

1. *If* $x>y$, *then* $(x \pm a)>(y \pm a)$.

For if $x>y$, $x-y$ is positive.

$\therefore$ $(x \pm a)-(y \pm a)$ is positive. Why?

But if $(x \pm a)-(y \pm a)$ is positive, then $(x \pm a)$ must be greater than $(y \pm a)$.

$\therefore$ $(x \pm a)>(y \pm a)$, and the sense of the inequality remains the same.

ILLUSTRATIONS. If $5>2$, then $(5 \pm 7)>(2 \pm 7)$.

For $12>9$, and $-2>-5$.

2. *If $x > y$, then $ax > ay$ and $\frac{x}{b} > \frac{y}{b}$ when a and b are positive.*

(*a*) For, if $x > y$, then $x - y$ is positive. $\therefore ax - ay$ is positive.

But if $ax - ay$ is positive, then ax must be $> ay$, and the sense remains unchanged.

(*b*) If $x > y$, then $x - y$ is positive and $\frac{x}{b} - \frac{y}{b}$ is positive.

But if $\frac{x}{b} - \frac{y}{b}$ is positive, then $\frac{x}{b}$ must be $> \frac{y}{b}$, and the sense remains unchanged.

ILLUSTRATION. If $5 > 2$, then $35 > 14$ and $\frac{5}{7} > \frac{2}{7}$. How were the last two obtained?

3. *If $x > y$ and $a > b$, then $x + a > y + b$ and $ax > by$ if all members are positive.*

(*a*) For, if $x > y$ and $a > b$, then $x - y$ is positive and $a - b$ is positive.

$\therefore (x + a) - (y + b)$ is positive. But if $(x + a) - (y + b)$ is positive, then $x + a$ must be $> y + b$, and the sense remains unchanged.

(*b*) If $x > y$ and $a > b$, then $ax > by$, when the members are positive.

For, since $x > y$, and a is positive, then $ax > ay$.

And since $a > b$, and y is positive, then $ay > by$.

But if $ax > ay$, and $ay > by$, then $ax > by$, and the sense is unchanged.

ILLUSTRATION. If $3 > 2$, and $5 > 4$, then $15 > 8$.

4. *If $x > y$, then $x^n > y^n$ when x and y are positive and n is a positive integer.*

This is but a special case of (3 *b*) just demonstrated. Why?

ILLUSTRATION. If $2 < 3$, then $2^3 < 3^3$, or $8 < 27$.

5. *If $x > y$, then $x^n > y^n$ when n is odd and a positive integer, and x and y are positive or negative.*

The case in which x and y are positive was demonstrated in (4).

If x and y are negative, let $x = -s$ and $y = -t$.

Then $-s > -t$ and $(-s)^n > (-t)^n$.

For $(-s)^n > (-t)^n \equiv -s^n > -t^n$.

The signs of the members are unchanged and the sense of the inequality remains the same.

ILLUSTRATION. If $-2 > -3$, then $(-2)^3 > (-3)^3$, or $-2^3 > -3^3$, or $-8 > -27$.

6. *If $x^n > y^n$, then $\sqrt[n]{x^n} > \sqrt[n]{y^n}$ or $x > y$, when n is an odd number.*

For an odd root of both members does not change the signs of the members and the sense remains the same.

ILLUSTRATION. If $27 < 64$, then $\sqrt[3]{27} < \sqrt[3]{64}$, or $3 < 4$.

7. *If $x^m < y^m$, then $\sqrt[m]{x^m} < \sqrt[m]{y^m}$ when m is even and positive roots are compared.*

For $\sqrt[m]{x^m} = \pm x$, and $\sqrt[m]{y^m} = \pm y$.

When $+x$ and $+y$ are compared, the inequality $+x < +y$ has the same signs as the original inequality and is in the same sense.

ILLUSTRATION. If $4 < 9$, then $\sqrt{4} < \sqrt{9}$, or $2 < 3$.

273. Prop. 2. The sense of an inequality is changed—

1. By changing the signs of both members.
2. By multiplying or dividing both members by the same negative number.
3. By raising both members to the same even power when both are negative.
4. By extracting the same even root of both members and comparing the negative roots.

Demonstration.

1. *If $x > y$, then $-x < -y$.*

For if x is numerically greater than y, then $-x$ is numerically greater than $-y$. But of two negative quantities the numerically greater one is the less.

$\therefore$ If $x > y$, then $-x < -y$.

ILLUSTRATION. If $6 > 4$, then $-6 < -4$.

2. *If $x > y$, then $-nx < -ny$.*

For if $x > y$, then $nx > ny$. Why?

If $nx > ny$, then $-nx < -ny$. Why?

$\therefore$ If $x > y$, then $-nx < ny$.

ILLUSTRATION. If $9 > 5$, then $-18 < -10$.

3. *If $-s > -t$, then $(-s)^n < (-t)^n$, or $s^n < t^n$, if n is even.*

For if n is even, $(-s)^n = +s^n$ and $(-t)^n = +t^n$.

Since the signs of the members have been changed, the sense has been changed in accordance with (1).

$\therefore$ If $-s > -t$, then $s^n < t^n$ when n is even.

ILLUSTRATION. If $-5 > -10$, then $25 < 100$.

4. *If $x^m > y^m$, then $\sqrt[m]{x^m} < \sqrt[m]{y^m}$ when m is even and the negative roots are compared.*

For $\sqrt[m]{x^m} = \pm x$, and $\sqrt[m]{y^m} = \pm y$. If the negative roots are compared, the signs of the members are changed, and in accordance with (1) the sense is changed.

$\therefore$ If $x^m > y^m$, then $\sqrt[m]{x^m} < \sqrt[m]{y^m}$, or $-x < -y$. When?

ILLUSTRATION. If $625 > 81$, then $\sqrt{625} < \sqrt{81}$, or $-25 < -9$, by comparing negative roots.

274. Transformation of inèqualities is the process of changing their form without changing their sense.

The **principal transformations** are clearing of fractions, transposition, collecting terms, division, raising to powers, and extracting roots.

ILLUSTRATION. $8 - \frac{8}{5} > \frac{16}{10} + 2;$

$$80 - 16 > 16 + 20, \text{ by clearing;}$$
$$64 > 36, \text{ by collecting;}$$
$$16 > 9, \text{ by dividing;}$$
$$4 > 3, \text{ by extracting root.}$$

275. Prop. 3. The sum of the squares of two unequal numbers is greater than twice the product of the two numbers.

Dem. Let a and b represent the two numbers and let $a \neq b$. Then $a^2 + b^2 > 2\,ab$.

Since $a \neq b$, then $(a - b) \neq 0$. Why?

Also $(a - b)^2 \neq 0$. Why?

And $(a - b)^2 \not< 0$. Why?

Then $(a - b)^2 > 0$. Why?

$\therefore\ a^2 - 2\,ab + b^2 > 0$, or

$a^2 + b^2 > 2\,ab$, by adding $2\,ab$ to both members.

MAXIMA AND MINIMA

276. If a variable quantity x cannot be greater than a constant quantity a but can approach indefinitely near it and even equal it in value, then is a the **Maximum Value** of x.

277. If a variable quantity x cannot be less than a constant quantity b but can approach indefinitely near it and even equal it in value, then is b the **Minimum Value** of x.

ILLUSTRATIONS. 1. In the expression $x = 8\,y - y^2$, x cannot be greater than 16, but is just equal to 16. Hence 16 is the maximum value of x. This may be shown as follows: Since y may have any value, assign the values 1, 2, 3, etc., in turn.

For $y=1$, $x=7$; for $y=2$, $x=12$; for $y=3$, $x=15$; for $y=4$, $x=16$; for $y=5$, $x=15$; for $y=6$, $x=12$; for $y=8$, $x=0$; for $y=3.9$, $x=15.99$; for $y=4.1$, $x=15.99$. $\therefore$ 16 is the maximum value of x.

2. Find the minimum value of x in the expression $x=y^2-6y+10$.

For $y=1$, $x=5$; for $y=2$, $x=2$; for $y=3$, $x=1$; for $y=4$, $x=2$. $\therefore$ the minimum value must be near 1.

For $y=2.09$, $x=1.8281$; for $y=3.01$, $x=1.0001$.

Since a value of y a little above 2 and one a little above 3 give a value of $x>1$, 1 must be the minimum value of x.

Model Solutions

1. Find the limit of $3x+\frac{x}{2}-12>\frac{5x}{2}+\frac{2}{3}$.

1.	$18x+3x-72>15x+4$.	Why?
2.	$18x+3x-15x>4+72$.	Why?
3.	$6x>76$.	Why?
4.	$x>12\frac{2}{3}$.	Why?

2. Prove that the minimum and maximum values of $x^3-3x^2-24x+85$ are 5 and 113 respectively.

Let $y=x^3-3x^2-24x+85$.

For $x=1$, $y=59$; for $x=2$, $y=33$; for $x=3$, $y=13$; for $x=4$, $y=5$; for $x=5$, $y=15$. $\therefore$ 5 is a minimum value of $x^3-3x^2-24x+85$.

For $x=0$, $y=85$; for $x=-1$, $y=105$; for $x=-2$, $y=113$; for $x=-3$, $y=103$. $\therefore$ 113 is a maximum. If these processes be continued, other minima and maxima may be found.

3. Divide $2a$ into two such parts that their product shall be a maximum.

1.	Let	$x=$ one part.
2.	Then	$2a-x=$ the other part.
3.	Let	$y=$ their product.
4.	Then	$(2a-x)x=y$.

5. $x^2 - 2ax + y = 0.$

6. $x = \dfrac{2a \pm \sqrt{4a^2 - 4y}}{2}$, by Art. **214**.

7. $x = a \pm \sqrt{a^2 - y}.$

That x shall be real, y must not be greater than a^2. Why?

8. $\therefore x = a \pm \sqrt{a^2 - a^2} = a$, the first part.

9. $2a - x \equiv 2a - a = a$, the second part.

10. $a \cdot a = a^2$, the maximum product.

4. Which is the greater, $2x^3$ or $x+1$, if $x>1$?

1. If $x>1$,
2. then $x^2>1$, Why?
3. and $2x^3>2x$; Why?
4. also $2x>x+1$. Why?
5. $\therefore 2x^3>x+1$. Why?

5. Given $\left\{\begin{array}{l}(1)\ 3x-4<x+6 \\ (2)\ 5x+7>3x+13\end{array}\right\}$, to find an integral value of x.

3. $3x - 4 < x + 6.$	7. $5x + 7 > 3x + 13.$		
4. $3x - x < 6 + 4.$	8. $5x - 3x > 13 - 7.$		
5. $2x < 10.$	9. $2x > 6.$		
6. $x < 5.$	10. $x > 3.$	$\therefore x = 4.$	

6. What values of x and y satisfy the inequalities:

$$\left\{\begin{array}{l}(1)\ 3x-y>12, \\ (2)\ -x+3y>2?\end{array}\right.$$

3. $9x - 3y > 36$ Why?
4. $-\ \ x + 3y > 2$
5. $8x \quad > 38$ Why?
6. $x \quad > 4\frac{3}{4}$ Why?
7. $3x - \ y > 12$
8. $-3x + 9y > 6$ Why?
9. $8y > 18$ Why?
10. $y > 2\frac{1}{4}$ Why?

7. Find the limits of x in $x^2+4x>45$.

1. $x^2+4x>45$.
2. $x^2+4x-45>0$. Why?
3. $(x-5)(x+9)>0$. Why?
4. $\therefore (x-5)(x+9)$ is positive. Why?
5. If $(x-5)(x+9)$ is positive, then the factors must be both positive or both negative. $x-5$ is positive when $x>5$. Both factors are negative when $x<-9$.

$\therefore x>5$ and $x<-9$, which means that x can have all values greater than 5 and less than -9; or, to put it negatively, x can have all values except 5 and -9 and intermediate values.

8. If a and b are positive integers and unequal, prove $a^3+b^3>a^2b+ab^2$.

1. $a^2+b^2>2ab$, by Prop. 3.
2. $a^2-ab+b^2>ab$. Why?
3. $(a+b)(a^2-ab+b^2)>ab(a+b)$. Why?
4. $a^3+b^3>a^2b+ab^2$.

EXAMPLES

Find the limits of x in the following:

1. $5x-6<3x+8$.
2. $\frac{7}{6}-\frac{5}{4}x<8-2x$.
3. $6x-\frac{2x-3}{4}>3x-\frac{4x-1}{2}$.
4. $\begin{cases}5x-y>20,\\ 2x-y<8.\end{cases}$
5. $\begin{cases}3x-2>\frac{1}{2}x-\frac{4}{5},\\ \frac{7}{6}-\frac{5}{4}x<8-2x.\end{cases}$
6. $7x-7\frac{2}{3}>\frac{2x}{3}+5$.
7. $\frac{a^2-b^2}{6d}>\frac{a^2-b^2}{3x}$.
8. $x^2-2x-15>0$.
9. $\begin{cases}2x+1<3x-3,\\ 5x-6<3x+8.\end{cases}$
10. $\begin{cases}\frac{1}{4}(x+2)+\frac{1}{3}x>\frac{1}{2}(x+1)+\frac{1}{3},\\ \frac{1}{4}(x+2)+\frac{1}{3}x<\frac{1}{2}(x-4)+3.\end{cases}$

If x, y, and z are positive and unequal, prove:

11. $x^3+y^3>xy(x+y)$.
12. $x^4+y^4>x^3y+xy^3$.
13. $x^3-y^3>3x^2y-3xy^2$.
14. $(y-z)^3<y^3-z^3$.
15. $\frac{1}{2}x^2+y^2+\frac{1}{2}z^2>xy+yz$.
16. $4x^2+9y^2-2xy>2a-(2a-10xy)$.

CHAPTER VIII

RATIO, PROPORTION, AND VARIATION

SECTION I

RATIO

278. Ratio is the relative magnitude of two numbers of the same kind. It is *found* by dividing one of the numbers by the other. It is *expressed* by writing one of the symbols of division between the numbers; as,

1. $\frac{a}{b}$. 2. $a \div b$. 3. $a : b$.

ILLUSTRATIONS. The ratio of 28 to 14 is $\frac{28}{14}$, or 2, which shows that 28 is twice as great as 14. The ratio of 3 monkeys to 12 monkeys is $\frac{3}{12}$, or $\frac{1}{4}$, which shows that 3 monkeys are $\frac{1}{4}$ of 12 monkeys. Why is there no ratio of 5 apples to 10 oranges? Of 6 to 3 men? Of 15 golf balls to 2 stymies?

279. The **Terms** of a ratio are the numbers compared, the first number being called the **first term, dividend, numerator,** or **antecedent**; the second being called the **second term, divisor, denominator,** or **consequent.**

Since a ratio may be considered a fraction, all the principles of fractions apply to, and hold good for, ratios. What operations on its terms multiply a ratio? State the reasons. What operations divide a ratio? State the reasons. What operations do not change the value of a ratio? State the reasons.

280. A **Direct Ratio** is the quotient of the first term divided by the second.

281. An **Inverse Ratio** is the reciprocal of the direct ratio.

ILLUSTRATION. If $\frac{a}{b}$ is a direct ratio, $\frac{b}{a}$ is its inverse ratio.

282. A **Compound Ratio** is the product of two or more simple ratios.

ILLUSTRATION. $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$, or $\frac{ace}{bdf}$, is a compound ratio. It is like what kind of fraction in arithmetic?

283. Two ratios are **compared** by reducing them to their Lowest Common Denominator and then comparing their numerators.

ILLUSTRATIONS. Is $\frac{11}{12} =, >,$ or $< \frac{13}{16}$?

L.C.D. $= 48$. $\frac{11}{12} = \frac{44}{48}$, and $\frac{13}{16} = \frac{39}{48}$. $\therefore \frac{11}{12} > \frac{13}{16}$.

284. An **Incommensurable Ratio** is one which cannot be expressed by two integers, as $\frac{\sqrt{2}}{\sqrt{3}}$, or $\frac{\sqrt[3]{2}}{2}$.

Numbers whose ratios can be exactly expressed are said to be **Commensurable**; all other numbers are said to be **Incommensurable.**

The ratio of two incommensurable numbers can be found to any desired degree of approximation; as,

Circumference $\div$ diameter $= \pi = 3.141592653589793 +$.

285. A ratio may be $=, >,$ or $<$ unity.

When is a ratio $= 1$? When > 1? When < 1? Which ratio shows equality? Which ratios show inequality?

286. A ratio of **Greater Inequality** is one which is $>$ unity.

ILLUSTRATION. $\frac{5}{3}$.

287. A ratio of **Less Inequality** is one which is $<$ unity, as $\frac{3}{5}$.

Questions. If 2 is added to both terms of $\frac{2}{3}$, is the value of the fraction changed? Why? What is the effect on the value? Does it cause $\frac{2}{3}$ to approach or recede from unity? What kind of inequality is $\frac{4}{5}$? What conclusion, then, may be drawn?

Put $\frac{5}{2}$ in place of the $\frac{2}{3}$ and answer the same questions.

What conclusion may be drawn from both of these examples?

If 3 is subtracted from both terms of $\frac{5}{8}$, is the value of the fraction increased or diminished? If 3 is subtracted from both terms of $\frac{8}{5}$, is the value of the fraction increased or diminished?

What is the effect on the value of a ratio of greater inequality of adding the same positive quantity to both of its terms? Of subtracting?

Answer the same questions when a ratio of less inequality is concerned.

288. Prop. A ratio of greater or less inequality is made more nearly equal to unity by the addition of a positive integral quantity to both of its terms.

Dem. Let $\frac{s}{t}$ represent any ratio $\neq 1$, and x any positive quantity. Then is

$$\frac{s+x}{t+x} \text{ more nearly equal to 1 than is } \frac{s}{t}.$$

To find how far these fractions are from unity, subtract 1 from each.

$$1.\ \frac{s}{t}-1=\frac{s-t}{t}. \qquad 2.\ \frac{s+x}{t+x}-1=\frac{s-t}{t+x}.$$

Now (2) is $<$ (1). Why?

$$\therefore\ \frac{s+x}{t+x} \text{ is nearer to, and } \therefore \text{ more nearly equal to, unity.}$$

For numerical illustrations, see the questions immediately following Art. **287**.

289. Cor. 1. A ratio of greater inequality is diminished by adding the same positive quantity to both of its terms, and *vice versa.*

How does this follow from the proposition?

290. Cor. 2. A ratio of less inequality is increased by . . . . (To be filled out by the student.)

How does this follow from the proposition?

291. Cor. 3. The limit of $\dfrac{s+x}{t+x}$, when x becomes very great, is unity.

Dem. Since the difference between $\dfrac{s+x}{t+x}$ and unity is $\dfrac{s-t}{t+x}$, this difference becomes very small if x is made very great, and in the limit is practically zero.

But if the difference between $\dfrac{s+x}{t+x}$ and unity is zero, then $\dfrac{s+x}{t+x}$ and unity must be of the same value.

$$\therefore\ \frac{s+x}{t+x}=1, \text{ when } x \text{ is very great.}$$

EXAMPLES

What is the ratio of:

1. 5 to 7?
2. 7 to 5?
3. $\frac{2}{3}$ to $\frac{8}{15}$?
4. $2\frac{5}{7}$ to $7\frac{88}{91}$?
5. a^3+b^3 to a^2-b^2?
6. $x-3$ to x^3-27?
7. a^6+b^6 to $a^4-a^2b^2+b^4$?
8. a^4+4b^4 to $a^2-2ab+2b^2$?
9. $x^3+2x^2-13x+10$ to $x^3+x^2-10x+8$?
10. $2x^4+x^3+2x^2+1$ to $3x^4+2x^3+3x^2+1$?
11. $\dfrac{x}{1+x}+\dfrac{1-x}{x}$ to $\dfrac{x}{x+1}-\dfrac{1-x}{x}$? Check.

12. $\frac{2}{3}(x^2 - \frac{1}{2}x - \frac{3}{2})$ to $\frac{3}{2}(x^2 + \frac{1}{3}x - \frac{2}{3})$? Check.

13. $\left(\frac{1}{a^2} + \frac{2}{ax} + \frac{1}{x^2} - 1\right)$ to $\left[\frac{x + a(1 - x)}{ax} \cdot \left(\frac{1}{x} + \frac{1}{a} + 1\right)\right]$?

14. $\frac{1 - \frac{1}{2}[1 - \frac{1}{3}(1 - x)]}{1 - \frac{1}{3}[1 - \frac{1}{2}(1 - x)]}$ to $\frac{3(\frac{1}{3}x^2 - x - 1) - 1}{4(\frac{1}{4}x^2 - x - 1) - 1}$? Check.

Find the compound ratio of:

15. $\frac{\left(1 + \frac{x}{a}\right)^3}{\left(1 - \frac{x}{a}\right)^3}$, $\frac{\left(\frac{x}{a} - 1\right)^2}{\left(\frac{a}{x} + 1\right)^2}$, and $\frac{\frac{a}{x}}{1 + \frac{2a}{x - a}}$. Check.

16. $\frac{1 - x^3}{(1 + x)^3}$, $\frac{3x^3 + 6x^2 + 3x}{4 - 4x}$, and $\frac{4x + 4}{3 + 3x + 3x^2}$.

17. $\frac{x^3 - 8a^3}{x^3 + 8a^3}$, $\frac{x^3 - 2x^2a + 4xa^2}{ax^2 + 2a^2x + 4a^3}$, and $\frac{x + 2a}{x - 2a}$.

Find the ratio of x to y if:

18. $5x = 10y$. **19.** $\frac{2}{3}x = \frac{8}{9}y$. **20.** $3x - 2y = 0$.

21. $x^2 - 4y^2 = 0$. **22.** $15x^2 - 10xy - 42y^2 = 7xy$.

23. Which is the greater, $\frac{17}{19}$, $\frac{37}{38}$, $\frac{46}{48}$, or $\frac{55}{57}$?

24. Which is the greater, $(a + x) \div (a^2 - ax + x^2)$, or $(a - x) \div (a^3 + x^3)$?

25. What number must be added to each term of the ratio $\frac{5}{14}$ to make the ratio $\frac{3}{4}$?

26. For what value of x will the ratio $(12 + x) : (25 + x) = \frac{2}{3}$?

27. Divide 42 into two parts which shall be to each other as $2 : 5$; $\frac{3}{5} : \frac{7}{9}$.

28. Find two numbers in the ratio of 8 to 13, whose sum is 147.

Section II

PROPORTION

292. Proportion is an equality of ratios, all the terms of the ratios being expressed.

Four numbers a, b, c, and d are in proportion when

$$\frac{a}{b}=\frac{c}{d}.$$

A proportion is *true* when the 1st term $\div$ the 2d $=$ the 3d $\div$ the 4th.

A proportion may be *expressed* in a number of ways, as:

1. $\frac{a}{b}=\frac{c}{d}$.
2. $a \div b = c \div d$.
3. $a : b :: c : d$.
4. $a : b = c : d$.

and *read* in a number of ways, as: "a is to b as c is to d"; "a is to b equals c is to d"; "a divided by b equals c divided by d."

293. The **Extremes** are the first and the fourth terms. The **Means** are the second and the third terms. The different terms are called **Proportionals.**

294. A **Continued Proportion** is a series of equal ratios in which the first term divided by the second equals the second divided by the third, etc., as $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\cdots$

295. If a, b, and c are in continued proportion, as $a:b::b:c$, then b is a **Mean Proportional** between a and c, and a is a **Third Proportional** to b and c, or c is to a and b.

296. A **Fourth Proportional** is the fourth term of a proportion. In $a : b :: c : x$, x is a fourth proportional to a, b, c.

297. A proportion is taken by **Inversion** when the reciprocals of the ratios are taken.

ILLUSTRATION. Taken by inversion, $\frac{a}{b} = \frac{c}{d}$ becomes $\frac{b}{a} = \frac{d}{c}$.

298. A proportion is taken by **Alternation** when the first and the fourth terms, or the second and the third terms, are made to change places.

ILLUSTRATION. Taken by alternation, $\frac{a}{b} = \frac{c}{d}$ becomes

1. $\frac{d}{b} = \frac{c}{a}$. 2. $\frac{a}{c} = \frac{b}{d}$.

299. A proportion is taken by **Addition** when the sum of the terms of each ratio is compared with either term of that ratio, the same order being observed in both ratios.

ILLUSTRATIONS. Taken by addition, $\frac{a}{b} = \frac{c}{d}$ becomes

$$\frac{a+b}{b} = \frac{c+d}{d}, \quad \frac{a}{a+b} = \frac{c}{c+d}, \quad \frac{a+b}{a} = \frac{c+d}{c}, \quad \text{or} \quad \frac{b}{a+b} = \frac{d}{c+d}.$$

300. A proportion is taken by **Subtraction** when the difference of the terms of each ratio is compared with either term of that ratio, the same order being observed in both ratios.

ILLUSTRATIONS. Taken by subtraction, $\frac{a}{b} = \frac{c}{d}$ becomes

$$\frac{a-b}{b} = \frac{c-d}{d}, \quad \frac{b-a}{b} = \frac{d-c}{d}, \quad \frac{a-b}{a} = \frac{c-d}{c}, \quad \frac{a}{a-b} = \frac{c}{c-d},$$

$$\frac{b}{a-b} = \frac{d}{c-d}, \text{ etc.}$$

301. A proportion is taken by **Addition and Subtraction** when the sum of the terms of each ratio is compared with their difference, the same order being observed in both ratios.

ILLUSTRATIONS. Taken by addition and subtraction, $\frac{a}{b}=\frac{c}{d}$ becomes $\frac{a+b}{a-b}=\frac{c+d}{c-d}$, $\frac{a-b}{a+b}=\frac{c-d}{c+d}$, $\frac{b+a}{b-a}=\frac{d+c}{d-c}$, or $\frac{b-a}{b+a}=\frac{d-c}{d+c}$.

302. Prop. 1. When four numbers are in proportion, the product of the extremes equals the product of the means.

Dem. Let $\frac{a}{b}=\frac{c}{d}$.

Then $ad=bc$, by clearing of fractions.

But ad is the product of the extremes, and bc is the product of the means. Hence the truth of the proposition appears.

303. COR. 1. If any three terms of a proportion are known, the fourth term is found by dividing the product of the given means (or extremes) by the given extreme (or mean).

Dem. Let $\frac{a}{b}=\frac{c}{d}$. Then $ad=bc$. Why?

And $a=\frac{bc}{d}$, $b=\frac{ad}{c}$, $c=\frac{ad}{b}$, $d=\frac{bc}{a}$. Why?

304. COR. 2. A mean proportional between two numbers is equal to the square root of their product.

Dem. Let $\frac{a}{x}=\frac{x}{d}$. Then $x=\sqrt{ad}$.

For $x^2=ad$. Why?

$\therefore x=\sqrt{ad}$. Why?

305. Prop. 2. If four numbers are in proportion, they are in proportion by *inversion.*

Dem. Let $\frac{a}{b}=\frac{c}{d}$. Then $\frac{b}{a}=\frac{d}{c}$.

For $1 \div \frac{a}{b}=1 \div \frac{c}{d}$, or

$$\frac{b}{a}=\frac{d}{c}.$$

306. Prop. 3. If four numbers are in proportion, they are in proportion by *alternation.*

Dem. Let $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a}{c}=\frac{b}{d}$, or $\frac{d}{b}=\frac{c}{a}$.

For $ad=bc$. Why?

Then $\frac{ad}{cd}=\frac{bc}{cd}$. Why? And $\frac{ad}{ab}=\frac{bc}{ab}$. Why?

$\therefore \frac{a}{c}=\frac{b}{d}$. Why? And $\frac{d}{b}=\frac{c}{a}$. Why?

307. Prop. 4. If four numbers are in proportion, they are in proportion by *addition.*

Dem. Let $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a+b}{b}=\frac{c+d}{d}$.

For $\frac{a}{b}+1=\frac{c}{d}+1$. Why?

$\therefore \frac{a+b}{b}=\frac{c+d}{d}$. Why?

Give the demonstration for

1. $\frac{a}{a+b}=\frac{c}{c+d}$, 2. $\frac{b}{a+b}=\frac{d}{c+d}$, 3. $\frac{a+b}{a}=\frac{c+d}{b}$.

308. Prop. 5. If four numbers are in proportion, they are in proportion by *subtraction*.

Dem. Let $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a-b}{b}=\frac{c-d}{d}$.

For $$\frac{a}{b}-1=\frac{c}{d}-1. \quad \text{Why?}$$

And $$\frac{a-b}{b}=\frac{c-d}{d}. \quad \text{Why?}$$

309. Prop. 6. If four numbers are in proportion, they are in proportion by *addition and subtraction*.

Dem. Let $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

For $$1. \quad \frac{a+b}{b}=\frac{c+d}{d}, \text{ by Prop. 4.}$$

And $$2. \quad \frac{a-b}{b}=\frac{c-d}{d}, \text{ by Prop. 5.}$$

$$\therefore \quad \frac{a+b}{a-b}=\frac{c+d}{c-d}, \text{ by dividing (1) by (2).}$$

Give the demonstration for

$$1. \quad \frac{a-b}{a+b}=\frac{c-d}{c+d}. \qquad 2. \quad \frac{b+a}{b-a}=\frac{d+c}{d-c}.$$

310. Prop. 7. In a series of equal fractions (or ratios) the sum of the numerators is to the sum of the denominators as any one numerator is to its denominator.

Dem. Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots.$$

Then $$\frac{a+c+e+\cdots}{b+d+f+\cdots}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots.$$

For, let $\frac{a}{b}=r$; then $\frac{c}{d}=r,\ \frac{e}{f}=r\cdots$.

Then $a=br,\quad c=dr,\quad e=fr.$ Why?

$a+c+e+\cdots=(b+d+f+\cdots)\,r.$ Why?

$$\therefore\ \frac{a+c+e+\cdots}{b+d+f+\cdots}=r=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots.$$ Why?

ILLUSTRATIONS.

$$\frac{1}{2}=\frac{3}{6}=\frac{7}{14}=\frac{1+3+7}{2+6+14}=\frac{11}{22}=\frac{7-3-1}{14-6-2}=\frac{3}{6}=\frac{1+7}{2+14}=\frac{8}{16}.$$

311. Prop. 8. If the product of two numbers is equal to the product of two other numbers, the former numbers may be made the extremes (or means), and the latter the means (or extremes).

Dem. Let $ad=bc.$

Then $\frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{d}$. Why?

If $ad=bc$, prove:

1. $a\div c=b\div d.$
2. $c\div d=a\div b.$
3. $b\div a=d\div c.$
4. $c\div a=d\div b.$
5. $b\div d=a\div c.$
6. $d\div b=c\div a.$
7. $d\div c=b\div a.$

MODEL SOLUTIONS

1. If $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$, prove that $\frac{l}{a}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}$.

1. $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}.$

2. $\frac{l^2}{a^2}=\frac{m^2}{b^2}=\frac{n^2}{c^2}=\frac{l^2+m^2+n^2}{a^2+b^2+c^2}.$ Why?

3. $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{\sqrt{l^2+m^2+n^5}}{\sqrt{a^2+b^2+c^2}}.$ Why?

2. If $\frac{x}{b+c}=\frac{b}{x+c}=\frac{c}{x+b}$, show $x=\frac{b+c}{2}$; also $x=b=c$.

1. $$\frac{x}{b+c}=\frac{x+b+c}{2x+2b+2c}=\frac{1}{2}.\quad \text{Why?}$$

$$\therefore\ x=\frac{b+c}{2}.\quad \text{How?}$$

2. $$\frac{x}{x+b+c}=\frac{b}{x+b+c}=\frac{c}{x+b+c}.\quad \text{Why?}$$

$$\therefore\ x=b=c.\quad \text{Why?}$$

3. If $\frac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=\frac{m+n}{m-n}$, show that $\frac{\sqrt{x-a}}{\sqrt{x+a}}=\frac{n}{m}$.

1. $\frac{2\sqrt{x-a}}{2\sqrt{x+a}}=\frac{2n}{2m}$, by taking proportion by addition and subtraction.

2. $$\frac{\sqrt{x-a}}{\sqrt{x+a}}=\frac{n}{m}.$$

EXAMPLES

1. If $3a=4b$, find the ratio of a to b.
2. If $15x=35y$, find the ratio of x to y.
3. $ax-by=cy-dx$; find the ratio of x to y.
4. $x^2-2xy+y^2=0$; find the ratio of x to y.
5. $6x^2+5xy-6y^2=0$; find the ratio of x to y.
6. $\frac{x}{y}=\frac{x+y}{z}$; find the ratio of x to y in terms of y and z.
7. If $\frac{m}{x}=\frac{n}{y}$, is $\frac{n}{m}=\frac{y}{x}$? Why? Is $\frac{x}{y}=\frac{m}{n}$? Why?
8. If $\frac{ax}{l}=\frac{by}{m}=\frac{cz}{n}$, show that $\frac{cz}{n}=\frac{\sqrt{ax^2+by^2+cz^2}}{\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}}$.
9. If $\frac{\sqrt{x}+16}{\sqrt{x}+4}=\frac{\sqrt{x}+32}{\sqrt{x}+12}$, show that $\frac{\sqrt{x}+16}{\sqrt{x}+4}=2$.

10. If $\dfrac{\sqrt{x}+16}{\sqrt{x}+4}=2$, show that $\dfrac{\sqrt{x}+4}{12}=1$.

11. If $\dfrac{w}{x}=\dfrac{y}{z}$, prove $\dfrac{w+x}{x}=\dfrac{y+z}{z}$.

12. If $a:b=r:s$, is $a\div(a-b)=r\div(r-s)$?

13. If $\dfrac{a}{b}=\dfrac{c}{d}$, prove $\dfrac{a^2+c^2}{b^2+d^2}=\dfrac{ac}{bd}$.

14. If $\dfrac{a}{b}=\dfrac{c}{d}$, show that $\dfrac{xa+yb}{as+bt}=\dfrac{cx+dy}{cs+dt}$.

15. If $\dfrac{a}{b}=\dfrac{c}{d}$, show that $\dfrac{a^2+b^2}{c^2+d^2}=\dfrac{(a+b)^2}{(c+d)^2}$.

16. $\dfrac{x}{b-c}=\dfrac{y}{c-a}=\dfrac{z}{a-b}$; find $x+y+z=$?

17. $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a-b}$; find the value of $x-y+z$.

18. Find a fourth proportional to a, b, and c.

19. Find a fourth proportional to $5x^2y$, x^3, and xy.

20. $\dfrac{a}{b}=\dfrac{x}{y}$; show $\dfrac{a^2+b^2}{x^2+y^2}=\dfrac{a^2-b^2}{x^2-y^2}$.

21. $\dfrac{ab-cd}{a-c}=\dfrac{ce-fb}{c-f}$; show that $\dfrac{ab-cd}{a-c}=\dfrac{fd-ae}{f-a}$.

22. $\dfrac{x^2}{a^2+x^2}=\dfrac{(c-x)^2}{b^2+(c-x)^2}$; show $\dfrac{x}{a}=\dfrac{c-x}{b}$.

23. $\dfrac{x}{a}=\dfrac{c-x}{b}$; show $\dfrac{x}{a}=\dfrac{c}{a+b}$ and $x=\dfrac{ac}{a+b}$.

24. $\dfrac{x}{y+z}=\dfrac{y}{z+x}=\dfrac{z}{x+y}$; prove $\dfrac{x}{y+z}=\dfrac{1}{2}$.

25. $\dfrac{a}{b}=\dfrac{x}{y}$; show that $a:b::\sqrt{3a^2+5x^2}:\sqrt{3b^2+5y^2}$.

26. $\dfrac{a}{b+c}=\dfrac{b}{a+c}=\dfrac{c}{a+b}$; show that $c=\dfrac{a+b}{2}$; also $a=b=c$.

27. $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$; show $\frac{a}{b}=\frac{c+e}{d+f}=\frac{ax-ey}{bx-fy}=\frac{ax-cy-ez}{bx-dy-fz}$.

28. If a, b, c, d are in continued proportion, show that

$$\left(\frac{a-b}{b-c}\right)^3=\frac{a}{d};\text{ also }\left(\frac{a-b}{b-c}\right)^2=\frac{a}{c}.$$

29. $\frac{a}{b}=\frac{c}{d}$ and $\frac{m}{n}=\frac{x}{y}$; prove $\frac{am}{bn}=\frac{cx}{dy}$; also $\frac{a+m}{b+n}=\frac{c+x}{d+y}$.

30. Formulate two principles from Ex. 29.

31. $\frac{a}{b}=\frac{x}{y}$, is $\frac{a^2}{b^2}=\frac{x^2}{y^2}$; also $\frac{a^n}{b^n}=\frac{x^n}{y^n}$; also $\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{x}}{\sqrt{y}}$; also $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{x}}{\sqrt[n]{y}}$?

32. Formulate two principles from Ex. 31.

33. $$\frac{ax}{l}=\frac{by}{m}=\frac{c}{n}=\frac{z}{p},\text{ show }\frac{c}{n}=\frac{\sqrt{ax^2+by^2+2\,cz}}{\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+2\,np}}.$$

34. If 2, 3, 4, 7 are four numbers not in proportion, what number must be added to each to make the results proportional?

35. Generalize Ex. 34.

36. Find a mean proportional between a and b; 2 and 8.

37. Find a third proportional to m^2 and n; 25 and 7.

38. Express the ratio of $8\frac{3}{11}$ to $1\frac{3}{11}$ by the ratio of two whole numbers.

Section III

VARIATION

312. Variation is a term applied to the consideration of quantities so related to each other that any change in one makes the other change in the same ratio; *i.e.*, the ratio between them is always a fixed or constant quantity.

ILLUSTRATION. Take the case of money on deposit in the savings department of a bank. Any change in the amount

of the deposit makes a corresponding change in the amount of interest. The constant involved is the fixed rate per cent paid on all deposits. The relations between principal, rate, and interest may be represented by

$$\frac{i}{p}=r,\ i=rp,\ \text{or}\ \frac{i}{r}=p.$$

313. A **Variable** is a quantity which, in any particular problem, may change its value. Variables are usually represented by the letters x, y, z, etc.

In the Illustration of Variation pick out one or more variables.

314. A **Constant** is a quantity which in any particular problem has a fixed value. Constants are usually represented by the letters a, b, c, etc., or the figures 1, 2, 3, etc.

In the Illustration of Variation pick out a Constant. Give another example.

315. One variable is said to be a **Function** of another when they are so related that if a value of one be given, the corresponding value of the other may be found.

ILLUSTRATION. The interest is some function of the principal.

316. The **Sign** of variation is $\propto$. It is read "varies as;" thus, $x \propto y$ is read "x varies as y." It means that as x increases or diminishes so y increases or diminishes. It means also that x and y are so related that their ratio, $\frac{x}{y}$, is a constant quantity. $\therefore\ \frac{x}{y}=a$.

317. Kinds of Variation.

1. *Direct Variation.* One quantity varies *directly* as another when any change in the latter makes the former change in the same ratio.

ILLUSTRATION. $x \propto y$ directly if any change in y makes x change in the same ratio. Interest varies directly as the principal, for any change in the latter makes the former change in the same ratio.

2. *Inverse Variation.* One quantity varies *inversely* as another when the first varies directly as the reciprocal of the second.

ILLUSTRATION. "x varies inversely as z" is written $x \propto \frac{1}{z}$.

It is learned in physics that, with a constant temperature, the volume of a gas varies inversely as the pressure, that is, $V \propto \frac{1}{P}$. Thus the volume of a certain gas will be 7 times as large when the pressure is $\frac{1}{7}$ as great.

If $V = 5$ at $P = 9$, what is V at $P = 54$?

3. *Joint Variation.* One quantity varies *jointly* as two others when the first varies as the product of the other two.

ILLUSTRATION. The dividends received by the owners of stock vary jointly as the product of each one's stock and the rate per cent declared.

"x varies as y and z" is written $x \propto yz$.

4. *Direct and Inverse Variation.* One quantity varies directly as a second and inversely as a third when it varies directly as the product of the second and the reciprocal of the third.

ILLUSTRATION. "x varies directly as y and inversely as z" is written $x \propto \frac{y}{z}$.

According to Newton's Law of Gravitation, $G \propto \frac{mn}{D^2}$, in which $G =$ gravity (or attraction), m and $n =$ the masses of the two bodies, and $D =$ distance. Make an example and apply this formula. Write the law in words.

318. Prop. 1. A variation may always be reduced to the form of an equation.

Dem.

1. *a.* Let $x \propto y$.

   b. Then $\frac{x}{y}$ is a constant ratio, by definition.

   c. Let a represent the constant ratio.

   d. Then $\frac{x}{y} = a$; or,

   e. $x = ay$, an equation.

2. *a.* Let $x \propto \frac{1}{z}$.

   b. Then $x = b \cdot \frac{1}{z}$. Why? Or,

   c. $xz = b$, an equation.

3. *a.* Let $x \propto yz$.

   b. Then $\frac{x}{yz} = c$. Why? Or,

   c. $x = cyz$, an equation.

4. *a.* Let $x \propto \frac{y}{z}$.

   b. Then $\frac{xz}{y} = d$. Why? Or,

   c. $xz = dy$, an equation.

319. Prop. 2. If $x \propto s$ and $y \propto t$, then $\frac{x}{y} \propto \frac{s}{t}$.

Dem. 1. $x = as$ and $y = bt$. Why?

2. $\frac{x}{y} = \frac{as}{bt}$. Why?

3. $\therefore \frac{x}{y} \propto \frac{s}{t}$. Why?

320. Prop. 3. If $x \propto y$ when z is constant, and if $x \propto z$ when y is constant, then $x \propto yz$ when both y and z vary.

Dem. The variation of x depends partly upon the variation of y and partly upon that of z. Suppose the variations of y and z take place separately, that each in turn produces its effect upon x, and that $x = a$ when $y = b$, when $z = c$.

1. Let z be constant while y changes to b; and a' be the value of x due to a partial change in x not equal to a.

a. Then $\frac{x}{y} = \frac{a'}{b}$, or $\frac{x}{a'} = \frac{y}{b}$. Why?

2. Let y be constant $(= b)$ while z changes to c.
Then x proceeds from the value a' to the value a, because both y and z have now changed to b and c respectively.

b. $\therefore \frac{a'}{a} = \frac{z}{c}$. Why?

c. $\frac{x}{a'} \cdot \frac{a'}{a} = \frac{y}{b} \cdot \frac{z}{c}$.

d. $\frac{x}{a} = \frac{yz}{bc}$.

e. $\frac{x}{yz} = \frac{a}{bc}$.

f. $\therefore x \propto yz$.

Model Solutions

1. If $x \propto y$ and $x = 8$ when $y = 15$, find x when $y = 10$.

1. $\frac{x}{y} = a$.
2. $\frac{8}{15} = a$.
3. $\frac{x}{y} = \frac{8}{15}$.
4. $\frac{x}{10} = \frac{8}{15}$.

$\therefore x = 5\frac{1}{3}$ when $y = 10$.

2. If R varies inversely as S, and $R = 7$ when $S = 3$, find R when $S = 2\frac{1}{3}$.

1. $R \propto \frac{1}{S}$.
2. $R = \frac{a}{S}$.
3. $7 = \frac{a}{3}$.
4. $a = 21$.
5. $R = \frac{a}{S}$.
6. $R = \frac{21}{2\frac{1}{3}}$.
7. $\therefore R = 9$ when $S = 2\frac{1}{3}$.

EXAMPLES

1. If $x \propto y$, and equals 3 when $y = 7$, find x when $y = 12$.

2. If $x \propto \frac{1}{y}$, and equals 15 when $y = 4$, find x when $y = 10$.

3. If $x \propto y$, and equals 8 when $y = 4$, find y when $x = 20$.

4. If $x \propto z$, and $y \propto \frac{1}{z}$, show that $x \propto \frac{1}{y}$.

5. If $x \propto y$, show that $\frac{x}{z} \propto \frac{y}{z}$ and $xz \propto yz$.

6. If $x \propto y$, show that $x^2 + y^2 \propto x^2 - y^2$.

7. If $x \propto yz$, and $y = 5$ when $x = 9$ and $z = 7$, what is the value of y when $z = 70$ and $x = 54$?

8. If $x \propto \frac{1}{y}$, and for $x = 6$, $y = 2$, what is the value of x for $y = 3$?

9. If $x + y \propto x - y$, prove that $x^2 + y^2 \propto xy$.

10. If $x^2 \propto y^2$, and $x = 3$ when $y = 4$, find y when $x = \frac{1}{\sqrt{3}}$.

11. x varies as y and z jointly; if $x = 2$ when $y = \frac{3}{5}$ and $z = \frac{10}{27}$, find z when $x = 54$ and $y = 3$.

12. If $y = p + q$, in which $p \propto q$ and $q \propto \frac{1}{x}$; and if when $x = 1$, $y = 6$; and when $x = 2$, $y = 5$; prove that $y = \frac{4}{3}x + \frac{14}{3x}$.

13. $S \propto t^2$, when f is constant; and $s \propto f$, when t is constant; also $2s = f$ when $t = 1$. Find the equation between f, s, and t.

14. X varies as the sum of two numbers, one of which varies as y and the other as y^2. If $x = 7$ when $y = 1$, and 39 when $y = 3$, what is the value of x in terms of y?

15. The base and altitude of a triangle being given, how is the area found? Show that if the base is constant, the area varies as the altitude; if the altitude is constant, that the area varies as the base; and if the area is constant, that the altitude and the base vary inversely.

SYNOPSIS FOR REVIEW, CHAPTER VIII

RATIO, PROPORTION, AND VARIATION

- SECT. I **Ratio**
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  - Prop. Brought nearer to unity, **288**. Dem.
  - Cors.
    1. Ratio >1 diminished or increased, **289**.
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- SECT. II **Proportion**
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  - Props.
    1. Product of extremes = Product of means, **302**. Dem.
       - *Cor.* 1. Three terms known, fourth can be found, **303**. Dem.
       - *Cor.* 2. Mean proportional = square root of product of the other two, **304**. Dem.
    2. In proportion by inversion, **305**. Dem.
    3. In proportion by alternation, **306**. Dem.
    4. In proportion by addition, **307**. Dem.
    5. In proportion by subtraction, **308**. Dem.
    6. In proportion by addition and subtraction, **309**. Dem.
    7. In series of equal fractions, sum of numerators divided by sum of denominators = any one of the fractions, **310**. Dem.
    8. Product of two numbers = product of two others, **311**. Dem.
- SECT. III **Variation**
  - Variation, **312**; Variable, **313**; Constant, **314**; Function, **315**; Sign of Variation, **316**.
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    1. Variations reduced to equations, **318**. Dem.
    2. If $x \propto s$, and $y \propto t$, then $\dfrac{x}{y} \propto \dfrac{s}{t}$, **319**. Dem.
    3. $x \propto yz$ when y and z vary, and in turn are constants, **320**. Dem.

Sample Test Questions

1. Define ratio, proportion, direct ratio, inverse ratio, compound ratio.
2. Define incommensurable ratio, incommensurable numbers, proportion.
3. When is ratio $=, >, < 1$? Define ratio of greater inequality; of less inequality.
4. How is a ratio multiplied? Divided? State the reasons.
5. How is a ratio > 1 increased? Diminished? A ratio < 1? Why?
6. Define mean, third, fourth proportional; continued proportion.
7. When is a proportion said to be taken by inversion? By alternation? By addition and subtraction? When is a proportion true?
8. State the proposition, "In a series of equal ratios," etc., and give the demonstration.
9. Prove the right to take a proportion by inversion.
10. Prove the right to take a proportion by alternation.
11. Prove the right to take a proportion by addition.
12. Prove the right to take a proportion by subtraction.
13. Prove the right to take a proportion by addition and subtraction.
14. Define variation, variable, constant, function.
15. Define direct variation, inverse variation.
16. Define joint variation, direct and inverse variation.
17. Prove that variations may be reduced to equations.
18. Of what use is the study of variation?
19. Give Newton's Law of Gravitation in words; in symbols.
20. How does the volume of a gas vary? Give an illustration.
21. Ada divided 252 sweet pea blossoms between Ethel and Margaret in the ratio of $\frac{28}{75}$ to $\frac{7}{15}$. How many blossoms did each girl receive?
22. Esther's age is to Grace's as 2 to 3. If 18 years are added to Esther's and 2 years to Grace's, their ages will be as 3 to 2. Find the age of each.
23. Divide the number 21 into two such parts that the quotient of the less divided by the greater shall be to the quotient of the greater divided by the less as 4 to 25.
24. Solve $\dfrac{x-5}{x+5} = \dfrac{210}{307}$ by the principles of proportion.
25. Solve $\dfrac{x+a}{x-a} = \dfrac{b-c}{b+c}$ by the principles of proportion.
26. State the propositions used in solving Ex. 25.
27. When can the principles of proportion be used to advantage in solving equations?
28. Why is the subject of Proportion placed immediately after that of Fractions?

CHAPTER IX

FRACTIONAL EQUATIONS

321. A **Fractional Equation** is one in which one or more of the unknown quantities appear in the denominator.

ILLUSTRATION. $\dfrac{ax}{b} - \dfrac{c}{x-a} = \dfrac{b}{x+b}$.

322. Prop. 1. Multiplying both members of an equation by an expression containing the unknown quantity introduces new roots.

Dem. Let $x = a$ be the given equation.

Then $x^2 - x = ax - a$, by multiplying both members by $x - 1$.

$$x^2 - x - ax + a = 0. \quad \text{Why?}$$

$$(x-1)(x-a) = 0. \quad \text{Why?}$$

$$x = a. \quad \text{Why?}$$

$$x = 1, \text{ the root introduced.}$$

323. COR. Dividing both members of an equation by an expression containing the unknown quantity diminishes the number of roots.

Dem.
1. Let $x^2 = ax$ be the given equation.
2. Then $x^2 - ax = 0$.
3. $x(x - a) = 0$.
4. $\therefore\ x = a$, 0, the two roots.

But dividing both members of (1) by x,

5. $$\frac{x^2}{x}=\frac{ax}{x}.$$

6. $$x=a, \text{ one root.}$$

Hence, one root was lost by dividing by an expression containing the unknown quantity.

324. Prop. 2. From every fractional equation an integral equation can be deduced; but it is possible that roots not belonging to the problem may be introduced in the process of solution.

Dem. Let $m=n$ be a fractional equation and L the L. C. D. Clearing of fractions by multiplying both members by the L. C. D. gives $Lm=Ln$, or $Lm-Ln=0$, an integral equation. But since $m=n$ is a fractional equation, the L. C. D. contains the unknown quantities, and it is possible that some of the roots of $L=0$ may satisfy $Lm-Ln=0$ and still be entirely foreign to $m=n$.

Solve: 1. $$x+1+\frac{x^2-x-6}{x+2}=\frac{x+1}{x-5}.$$

2. $$(x+1)(x+2)(x-5)+(x^2-x-6)(x-5)$$
$$=(x+1)(x+2), \text{ by multiplying (1) by } (x+2)(x-5).$$

This equation is integral and is satisfied by any root of (1). It is necessary, however, to see whether any root of $(x+2)(x-5)=0$ satisfies (2). Solving $(x+2)(x-5)=0$, $x=-2, +5$.

Substituting $+5$ in (2), $0\equiv 42$, which is evidently not true. Hence $+5$ is not a root of (2). Substituting -2 in (2), $0\equiv 0$. Hence -2 is a root of (2). But it must now be tested in (1). If -2 is substituted in (1) as it now stands, $-2+1+\frac{0}{0}=\frac{-1}{-7}$, in which the indeterminate form $\frac{0}{0}$ appears.

Since $\dfrac{x^2 - x - 6}{x + 2} \equiv \dfrac{(x + 2)(x - 3)}{x + 2} \equiv x - 3$, (1) may be written

$$x + 1 + x - 3 = \frac{x + 1}{x - 5}.$$

Substituting -2, $\quad -2 + 1 - 2 - 3 = \dfrac{-1}{-7}$, or

$$-6 \equiv \tfrac{1}{7}, \text{ which is evidently not true.}$$

Hence -2 is not a root of (1), and was introduced into (2) during the process of obtaining an integral equation from the fractional one.

Note. $x = -2$ satisfies (2) because the numerator $x^2 - x - 6$ contains a factor, $x + 2$, of the L. C. D. All fractions, therefore, should be reduced to their lowest terms before clearing the equation of fractions.

325. Cor. In general, if both members of an equation are multiplied by the L. C. D., no roots will be introduced, because the equation contains fractions whose denominators are factors in the L. C. D. and are not in the numerator.

1. $$x + 1 + \frac{x^2 - 5x + 6}{x + 2} = \frac{x + 1}{x - 5}.$$

2. $(x + 1)(x + 2)(x - 5) + (x^2 - 5x + 6)(x - 5)$
$= (x+1)(x+2)$, by multiplying (1) by $(x + 2)(x - 5)$.

From $\quad (x + 2)(x - 5) = 0, \; x = -2, \; +5.$

If -2 and $+5$ are in turn substituted in (2), it is found that neither is a root of (2), and hence no new roots have been introduced.

Model Solutions

1. $\dfrac{6x + 7}{15} - \dfrac{2(x - 1)}{7x - 6} = \dfrac{2x + 1}{5}$. Solve and verify.

1. $-\dfrac{2(x - 1)}{7x - 6} = \dfrac{6x + 3}{15} - \dfrac{6x + 7}{15} = -\dfrac{4}{15}$. Why?
2. $30(x - 1) = 28x - 24$. Why?
3. $2x = 6$.
4. $x = 3$.

Verification

1. $\dfrac{18+7}{15}-\dfrac{2(3-1)}{21-6}=\dfrac{6+1}{5}.$

2. $\frac{25}{15}-\frac{4}{15}=\frac{7}{5}.$

3. $\frac{7}{5}\equiv\frac{7}{5}.$

2. $\dfrac{x-\dfrac{x-1}{3}}{3}+\dfrac{\dfrac{31}{9}}{4}=\dfrac{3-\dfrac{x-2}{4}}{5}.$ Solve and verify.

1. $\dfrac{3x-x+1}{9}+\dfrac{31}{36}=\dfrac{12-x+2}{20}.$

2. $\dfrac{8x+35}{36}=\dfrac{14-x}{20}.$

3. $\dfrac{8x+35}{9}=\dfrac{14-x}{5}.$

4. $40x+175=126-9x.$

5. $49x=-49.$

6. $x=-1.$

Verification

1. $\dfrac{-1-\dfrac{-1-1}{3}}{3}+\dfrac{\dfrac{31}{9}}{4}=\dfrac{3-\dfrac{-1-2}{4}}{5}.$

2. $-\frac{1}{3}+\frac{2}{9}+\frac{31}{36}=\frac{3}{5}+\frac{3}{20}.$

3. $\frac{27}{36}=\frac{15}{20}.$

4. $\frac{3}{4}\equiv\frac{3}{4}.$

3. $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+20}{x+4}=\dfrac{x^2+4x+6}{x+2}+\dfrac{x^2+6x+12}{x+3}.$

1. $x+1+\dfrac{1}{x+1}+x+4+\dfrac{4}{x+4}=x+2+\dfrac{2}{x+2}$
$+x+3+\dfrac{3}{x+3}.$

2. $\dfrac{1}{x+1}+\dfrac{4}{x+4}=\dfrac{2}{x+2}+\dfrac{3}{x+3}.$

3. $$\frac{5x+8}{x^2+5x+4}=\frac{5x+12}{x^2+5x+6}=\frac{4}{2}=2.\quad \text{Why?}$$

4. $2x^2+10x+8=5x+8.$ Why?

5. $2x^2+5x=0.$

6. $x(2x+5)=0.$

7. $x=0,\ -\frac{5}{2}.$ Why?

Verification

For $x=0$.

1. $$\frac{0+0+2}{0+1}+\frac{0+0+20}{0+4}=\frac{0+0+6}{0+2}+\frac{0+0+12}{0+3}.$$

2. $2+5=3+4.$

3. $7\equiv 7.$

For $x=-\frac{5}{2}$.

1. $$\frac{\frac{25}{4}-5+2}{-\frac{5}{2}+1}+\frac{\frac{25}{4}-20+20}{-\frac{5}{2}+4}=\frac{\frac{25}{4}-10+6}{-\frac{5}{2}+2}+\frac{\frac{25}{4}-15+12}{-\frac{5}{2}+3}.$$

2. $$\frac{25-12}{-10+4}+\frac{25}{-10+16}=\frac{25-16}{-10+8}+\frac{25-12}{-10+12}.$$

3. $$\frac{13}{-6}+\frac{25}{6}=\frac{9}{-2}+\frac{13}{2}.$$

4. $2\equiv 2.$

4. $\dfrac{2x^2-5x-6}{2x^2-7x+1}=\dfrac{2x^2-14x-12}{2x^2-16x+4}.$ Solve by Principles of Proportion.

1. $$\frac{2x^2-5x-6}{2x-7}=\frac{2x^2-14x-12}{2x-16}=\frac{9x+6}{9}.\quad \text{Why?}$$
$$=\frac{3x+2}{3}.\quad \text{Why?}$$

See Arts. **308, 310**.

2. $6x^2-15x-18=6x^2-17x-14.$ Why?

3. $2x=4.$

4. $x=2.$

Verification

1. $$\frac{8-10-6}{8-14+1}=\frac{8-28-12}{8-32+4}.$$

2. $\frac{8}{5}\equiv\frac{8}{5}.$

5. $\dfrac{x-a}{ab-ax}+\dfrac{x-a}{bc-bx}=\dfrac{x-a}{ac-ax}$. Solve and verify.

1. $\dfrac{1}{ab-ax}+\dfrac{1}{bc-bx}=\dfrac{1}{ac-ax}$,

and $x-a=0$, or $x=a$. Why?

2. $bc-bx+ab-ax=b^2-bx$.

3. $$x=\frac{ab+bc-b^2}{a}.$$

Verification

For $x=a$.

$$\frac{0}{ab-a^2}+\frac{0}{bc-ab}=\frac{0}{ac-a^2}, \text{ or } 0\equiv 0.$$

For $x=\dfrac{ab+bc-b^2}{a}$.

1. $$\frac{1}{ab-ab-bc+b^2}+\frac{1}{bc-\dfrac{ab^2+b^2c-b^3}{a}}=\frac{1}{ac-ab-bc+b^2}.$$

2. $$\frac{1}{b^2-bc}+\frac{a}{abc-ab^2-b^2c+b^3}=\frac{1}{ac-ab-bc+b^2}.$$

3. $$\frac{1}{b(b-c)}+\frac{a}{(b-c)(b-a)b}=\frac{1}{(b-c)(b-a)}.$$

4. $$\frac{1}{b}+\frac{a}{b(b-a)}=\frac{1}{b-a}.$$

5. $$\frac{1}{b-a}\equiv\frac{1}{b-a}.$$

Practical Suggestions

1. It is not always best to clear of fractions immediately. Solution No. 1.

2. It is sometimes best to transpose and unite some terms before clearing of fractions. Solution No. 1.

3. Complex fractions should be simplified. Solution No. 2.

4. It is often convenient to reduce improper fractions to mixed numbers. Solution No. 3.

5. It is often convenient partially to clear of fractions (as might have been done in Solution No. 1 by multiplying by 15).

6. Use the Principles of Proportion whenever convenient. Solution No. 4.

7. All fractions should be reduced to lowest terms before clearing of fractions. Note to Prop. 2.

8. Watch for terms which destroy each other, and for factors which can be removed from both members of the equation. Solution No. 5. Third step in No. 2.

9. Place all factors containing the unknown quantity, which have been removed from both members of the equation, equal to zero, and solve. Solution No. 5.

10. Observe the negative sign before any fraction.

11. Clear of fractions by multiplying by the L. C. D., so as not to introduce new roots.

12. In verifying the roots, do not use the process that was employed in solving for fear of making like mistakes.

13. In verifying literal roots, where the process is complicated, substitute numbers for letters.

EXAMPLES

Solve and verify:

1. $\frac{x+1}{x+2}-\frac{x+2}{x+3}=\frac{x+5}{x+6}-\frac{x+6}{x+7}$.

2. $\frac{8x+23}{20}-\frac{5x+2}{3x+4}=\frac{2x+3}{5}-1$.

3. $\frac{7-x}{x^2+x-2}=-\frac{7x-49}{7(x-3)(x-4)}$.

4. $\frac{4x+3}{9}=\frac{8x+19}{18}-\frac{7x-29}{5x-12}$.

5. $\frac{2x+5}{21}-\frac{x-15}{3\frac{1}{2}x}=\frac{x(4-2\frac{2}{3})}{14}$.

6. $\frac{3y+b-a}{2y+a-b}=\frac{6y-a+b}{4y-b+a}$.

7. $\frac{2v-3}{v-4}+\frac{3v-2}{v-8}=\frac{5v^2-29v-4}{v^2-12v+32}$.

8. $\frac{3}{4}+\frac{12-11x}{8x+20}-\frac{5x+2}{12x+30}=\frac{8-x}{6x+15}-\frac{1}{3}$.

9. $\left(\frac{x-m}{x-n}\right)^2=\frac{x-2m}{x-2n}.$

10. $\frac{3}{s-1}-\frac{s+1}{s-1}=\frac{s^2}{1-s^2}.$

11. $\frac{x+3}{\frac{1}{2}x+1}=\frac{2x+5}{2x+4}+\frac{\frac{3}{2}x+3}{\frac{3}{2}x+1}.$

12. $.5-\frac{3.5x}{x-5}-\frac{24-3x}{8}=.375x.$

13. $\frac{x+a}{x-a}-\frac{x-a}{x+a}-\frac{x^2}{a^2-x^2}-1=0.$

14. $\frac{x^2+3x}{x^2+3x+2}+\frac{4}{3x^2+6x}-1=0.$

15. $\frac{x-a}{2x-b}-\frac{3x-c}{6x-d}=0.$

16. $\frac{3x-1}{2x-1}-\frac{4x-2}{3x-1}=\frac{1}{6}.$

17. $\frac{x+a}{x-a}-\frac{x-b}{x+b}=\frac{2(a+b)}{x}.$

18. $\frac{6x+8}{2x+1}-\frac{2x+38}{x+12}-1=0.$

19. $\frac{ax^2+bx+c}{ax+b}=\frac{px^2+qx+r}{px+q}.$

20. $\frac{3x+1}{2x+3}-\frac{x+2}{6x+9}+\frac{1}{2x}=\frac{4}{3}.$

21. $\frac{a^3-a^2+a}{x^2+2x+4}\cdot\frac{x^3-8}{a^3+1}=\frac{\frac{a}{b}-\frac{b}{a}}{\frac{1}{b}-\frac{1}{a}}.$

22. $\frac{x-\frac{1}{2}}{x-1}-\frac{3}{5}\left(\frac{1}{x-1}-\frac{1}{3}\right)=\frac{23}{10(x-1)}.$

23. $\frac{3x-7}{2x-9}-\frac{3x-1}{2(x+5)}=\frac{\frac{45}{2}x-24}{2x^2-3x-27}.$

24. $\frac{ax-b}{ax+b}-\frac{bx-a}{bx+a}=\frac{a-b}{(ax+b)(a+bx)}.$

25. $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}=\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}.$

26. $\frac{2\frac{3}{4}x-1}{8\frac{3}{4}}-\frac{1\frac{1}{6}x-13\frac{1}{2}}{7\frac{1}{2}}-\frac{1\frac{2}{3}x-2}{10\frac{1}{2}}=\frac{x+46}{x-46}.$

27. $.15z+\frac{.135z-.225}{.6}=\frac{.36}{.2}-\frac{.09z^2-.18z}{.9z}.$

28. $\frac{a^2b^3}{a^2b(ab+x)-b^2(b^2-x)}=\frac{a^2b^2}{b^2(x-b)-a^2(ab-x)}.$

29. $\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{x^2-ab}.$

30. $1-x-\frac{1+x}{1-x}+\frac{3+2x^2}{2x-2}=0.$

31. $\frac{m(x+a)}{x+b}+\frac{n(x+b)}{x+a}=m+n.$

32. $\frac{x-2}{x-3}-\frac{x-3}{x-4}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$

33. $\frac{75-x}{3x+3}-5=\frac{23}{x+1}-\frac{80x+21}{15x+10}.$

34. $\frac{3-x}{1-x}-\frac{5-x}{7-x}=1-\frac{x^2-2}{7-8x+x^2}.$

35. $\frac{5(\frac{1}{4}+x)}{3x+1}-\frac{x-17}{2x+1}=\frac{73}{4}-\frac{10x}{1+3x}.$

36. $\frac{1}{a(c-x)}-\frac{1}{b(c-x)}-\frac{1}{a(b-x)}=0.$

37. $\frac{x+17}{x+20}-\frac{x+20}{x+23}=\frac{x-7}{x-10}-\frac{x-10}{x-13}.$

38. $\frac{x^3+2a^3}{x+a}+\frac{x^3-2a^3}{a-x}=\frac{a^3}{x^2-a^2}-2ax.$

39. $\frac{15x-7}{4+5x}-\frac{10x+3}{5x-4}+\frac{418}{25x^2-16}=1.$

40. $\frac{4x-15}{x-\frac{7}{2}}-\frac{x-4\frac{1}{4}}{\frac{1}{4}x-1}+\frac{4-5x}{1-x}-\frac{6\frac{1}{2}-5x}{1\frac{1}{2}-x}.$

41. $\frac{1}{2}x-\frac{\frac{2x-3}{3}-\frac{3x-1}{4}}{\frac{x-1}{2}}=\frac{3}{2}\cdot\frac{x^2-\frac{x}{3}+2}{3x-2}.$

42. $\frac{x}{2}-\frac{\frac{1}{3}(2x-3)-\frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)}=\frac{3}{2}\left(\frac{x^2+2}{3x-1}\right).$

43. $\frac{c^2+x^2}{cx}+\frac{c(x-c)}{x(x+c)}+\frac{cx}{x^2-c^2}-\frac{x(x+c)}{c(x-c)}=-2.$

44. $\frac{a-b}{x-c}+\frac{b-c}{x-a}+\frac{c-a}{x-b}=\frac{a+b}{x-a}+\frac{b+c}{x-b}-\frac{a+2b+c}{x-c}.$

PROBLEMS INVOLVING FRACTIONAL EQUATIONS, PROPORTION, AND VARIATION

1. What is the price of golf balls, if a reduction of 20% in price enables a player to buy 6 more of them for five dollars?

2. Divide 91 into two such parts that the quotient of the greater part divided by the difference between the parts may be seven.

3. If the hour and minute hands of a watch are together at 12 o'clock, at what time between 1 and 2 o'clock will they be together? Between 3 and 4? 7 and 8? 11 and 12? n and $n+1$? From the answer to the first, how may all the others be obtained?

4. At what time between 9 and 10 o'clock are the hands of a clock together? At right angles? Opposite?

5. What number must be subtracted from m and n so that the differences shall be in the ratio of r to s?

6. A cistern can be filled by two pipes running together, in 27 minutes. The larger pipe would fill the cistern in 54 minutes less than the smaller one. Find the time taken by each, carried out to three decimal places. What difference does it make in the solution whether the unknown quantities are represented by x and $x+54$, or x and $x-54$?

7. The sum of two numbers is 76, and their sum is to the first as $19:15$. What are the numbers?

8. The force of gravity varies inversely as the square of the distance from the center of the earth. At the distance 1 from the center of the earth, this force is expressed by the number 32.16. By what is it expressed at the distance 50?

9. The amount of water which flows from an orifice is jointly proportional to the area of the orifice and the velocity of the water. If water flows at the rate of $8:7$ through two orifices whose areas are as $5:13$, how much does each orifice discharge in a given time, provided the first discharges 561 cu. ft. more than the second?

10. A and B have the same amount of work to do. A began to work 2 hours before B, but afterward they worked and rested together until B had finished his work, which was 1 hour before A had finished his. If, when A had done half his work, each had taken the other's remaining work, B would have finished 2 hours and $1\frac{1}{2}$ minutes before A. How long did each take to do the work?

11. A hare, 95 of her leaps ahead of a hound, takes 5 leaps to the hound's 4; but 3 of the hound's leaps equal 4 of the hare's. How many leaps must the hound take to catch the hare?

12. The volume of a pyramid varies jointly as its base and altitude. A pyramid whose base is 9 feet square, and height 10 feet, contains 10 cubic yards. What must be the height of a pyramid with a base 3 feet square in order that it may contain 2 cubic yards?

13. When a body falls from rest, its distance from the starting point varies as the square of the time occupied in falling. If a ball falls 402.5 feet in 5 seconds, how far will it fall in 10 seconds?

14. The area of a circle varies as the square of its radius. The area of a circle whose radius is 10 feet is 314.159265. Find the area of a circle whose radius is 12 feet.

15. The solidity of a sphere varies as the cube of its diameter. If an orange 2 inches in diameter is worth 5 cents, how much is that orange worth whose diameter is 4 inches?

16. Show that the area of a circle 2.5 feet in diameter is equal to the sum of the areas of two circles 1 foot 6 inches and 2 feet in diameter.

17. The volume of a sphere varies as the cube of its radius. When the radius is $3\frac{1}{2}$ feet the volume is $179\frac{2}{3}$ cubic feet; find the volume when the radius is 1 foot 9 inches.

CHAPTER X

SYSTEMS OF LINEAR EQUATIONS

SECTION I

TWO UNKNOWN QUANTITIES

For classification of Equations, Definitions, etc., see Chapter III.

326. A **System** of equations is a group of two or more independent equations which are satisfied by the same set, or sets, of values of the unknown quantities.

327. **Simultaneous Equations** are the equations of a system, and hence like symbols of the unknown quantity stand for the same thing in both.

ILLUSTRATION.

$$\left\{\begin{array}{ll} 1. & x+2y+2z=11, \\ 2. & 2x+y+z=7, \\ 3. & 3x+4y+z=14, \end{array}\right\}$$

are simultaneous equations, for at the same time that $x=1$ in the first equation, $x=1$ in the second and the third equations. Likewise, $y=2$ and $z=3$ in all of the equations.

328. The **Main Problem** which arises in connection with every system of equations of condition is to find a set, or sets, of values of the unknown quantities which shall render every equation of the system an identity, literal or numerical.

329. A system is **Satisfied** when the substitution of the set, or sets, of values obtained renders every equation of the system an identity.

330. The **Solution** of a system is the set, or sets, of values obtained; and, as in the case of equations of one unknown quantity, the solution may be either **Formal** or **Algebraic.**

331. A solution is **Determinate** when the number of roots is finite.

332. A solution is **Indeterminate** when the number of roots is infinite.

ILLUSTRATIONS. *Example* 1. $x+y=5$ and $2x-y=1$.

In this example, $x=2$, $y=3$, is a solution, and there is no other. Hence, the roots are finite and the solution is determinate.

Example 2. $x-y=7$.

In this example,

If $x=8$,	then $y=1$;	and $x=8$,	$y=1$	is a solution.	
If $x=9$,	" $y=2$;	" $x=9$,	$y=2$	"	"
If $x=100$,	" $y=93$;	" $x=100$,	$y=93$	"	"
If $x=a$,	" $y=a-7$;	" $x=a$,	$y=a-7$	"	"

Since a represents any quantity, there must be an infinite number of roots. Hence the solution is indeterminate.

333. Prop. 1. A solution is in general determinate when the number of independent equations is equal to the number of unknown quantities involved.

Dem. The proposition is nearly axiomatic. As seen in the preceding illustration, there are two distinct quantities sought, and there are two distinct conditions to show their relation. Hence their values can be obtained.

334. COR. 1. When the number of independent equations is less than the number of unknown quantities, the solution is in general indeterminate.

Dem. Let A represent the number of unknown quantities, and B the number of equations, when $B < A$. Then the solution is indeterminate. For when any values are given to the first $A-B$ unknown quantities, the B equations give a determinate solution by Prop. 1. But the number of values which can be given to $A-B$ is infinite. Hence there is an infinite number of solutions, and the solution is therefore indeterminate.

335. Cor. 2. If the number of independent equations is greater than the number of unknown quantities, there is, in general, no solution, and the system is said to be inconsistent.

Dem. Let A represent the number of unknown quantities, and B the number of equations when $B > A$. Then the system is inconsistent.

For the first A equations are determinate by Prop. 1, and a set of these values will not necessarily satisfy any one of the $B-A$ equations which were not used in the solution, and in general will not do so.

Exception. The roots of the first two equations of the system $x + y = 5$, $2x - y = 1$, and $7x - 2y = 8$, are $x = 2$, $y = 3$. These values satisfy the third equation, $7x - 2y = 8$.

Note. It may be observed from this example that two of the equations can be so transformed that the sum of the corresponding members shall always produce the third equation. Thus, by multiplying both members of $2x - y = 1$ by 3 and adding $x + y = 5$ we have the third equation, $7x - 2y = 8$.

336. Elimination is the process of combining the equations of a system, two at a time, so as to produce a new equation, or system, containing one less equation and one less unknown quantity. The unknown quantity that is caused to disappear is said to be *eliminated*.

337. There are **Five General Methods of Elimination**: (1) Addition or Subtraction, (2) Substitution, (3) Comparison, (4) Arbitrary Multipliers, (5) Cross-multiplication.

Note. Any one of these methods will solve a system of equations, but no one method is always the best. It is well for the student to have at his command at least the first three of these methods. The fifth method is recommended to those who intend to enter college, for it leads up to the important subject of Determinants.

338. PROBLEM 1. **To eliminate by addition or subtraction.**

Rule.—1. *Reduce the equations to the form* $ax + by = c$.

2. *Multiply each equation by the quotient of the L. C. M. of the coefficients of the letter to be eliminated, divided by the coefficient of that letter in the equation.*

3. *If the signs of the quantity to be eliminated are unlike, add the equations; if alike, subtract. Then solve.*

Dem.—1. Reducing the given equations to the form $ax+by=c$ is according to the first three principal transformations, *i.e.*, clearing of fractions, transposition, and collecting terms. These transformations, as has been shown, produce equivalent equations. Hence the solution of the transformed equations is the solution of the original equations, and the roots of the former are the roots of the latter.

2. The second step in the process produces equivalent equations.

3. The third step is legitimate according to the axiom, if equals be added to equals, the sums will be equal; and if equals be subtracted from equals, the differences will be equal. This process eliminates the unknown quantity, because a negative quantity destroys a numerically equal positive quantity.

MODEL SOLUTIONS

1. Solve $\begin{cases} (1)\ \frac{2}{3}x + \frac{3}{4}y = \frac{1}{12}, \\ (2)\ \frac{4}{5}x - \frac{6}{7}y = \frac{24}{35}. \end{cases}$

3.	$8x + 9y = 1,$	by clearing (1) of fractions.
4.	$28x - 30y = 24,$	by clearing (2) of fractions.
5.	$80x + 90y = 10,$	by multiplying (3) by 10.
6.	$84x - 90y = 72,$	by multiplying (4) by 3.
7.	$164x = 82,$	by adding (5) and (6).
8.	$x = \frac{1}{2},$	by division.
9.	$\frac{8}{2} + 9y = 1,$	by substituting (8) in (3).
10.	$y = -\frac{1}{3}.$	

Let the student verify both roots in equations (1) and (2).

2. Solve $\begin{cases} (1)\ ax + by = c, \\ (2)\ a'x + b'y = c'. \end{cases}$

3.	$a'ax + a'by = a'c,$	by multiplying (1) by a'.
4.	$a'ax + ab'y = ac',$	by multiplying (2) by a.
5.	$(ab' - a'b)y = ac' - a'c,$	by subtracting (3) from (4).
6.	$y = \dfrac{ac' - a'c}{ab' - a'b},$	by division.
7.	$ab'x + bb'y = b'c,$	by multiplying (1) by b'.
8.	$a'bx + bb'y = bc',$	by multiplying (2) by b.
9.	$(ab' - a'b)x = b'c - bc',$	by subtracting (8) from (7).
10.	$x = \dfrac{b'c - bc'}{ab' - a'b}.$	

Verification

1.	$\dfrac{ab'c - abc'}{ab' - a'b} + \dfrac{abc' - a'bc}{ab' - a'b} = c,$	by substituting (10) and (6) in (1).
2.	$\dfrac{a'b'c - a'bc'}{ab' - a'b} + \dfrac{ab'c' - a'b'c}{ab' - a'b} = c',$	by substituting (10) and (6) in (2).
3.	$\dfrac{ab'c - a'bc}{ab' - a'b} = c,$	by addition.
4.	$\dfrac{ab'c' - a'bc'}{ab' - a'b} = c',$	by addition.
5.	$c \equiv c,$	by division.
6.	$c' \equiv c',$	by division.

EXAMPLES

Solve by the method of addition or subtraction, and verify:

1. $\begin{cases} 12x + 15y = 8, \\ 16x + 9y = 7. \end{cases}$

2. $\begin{cases} 7x - 9y = 13, \\ 5x + 2y = 10. \end{cases}$

3. $\begin{cases} 13x + 15y = 7, \\ 91x - 45y = 3. \end{cases}$

4. $\begin{cases} 27s - 6t = 9, \\ 18s + 15t = 17. \end{cases}$

5. $\begin{cases} 5v + 7w - 43 = 0, \\ 9w + 11v - 69 = 0 \end{cases}$

6. $\begin{cases} 18x - 20q = 13, \\ 16x + 17q = 40. \end{cases}$

7. $\begin{cases} 29s - 14z = 175, \\ 87s - 56z = 497. \end{cases}$

8. $\begin{cases} 69p - 17q = 103, \\ 14p - 13q = -41. \end{cases}$

9. $\begin{cases} 6x + 4y = 236, \\ 3x + 15y = 573. \end{cases}$

10. $\begin{cases} -5x - 9y = -188, \\ 13x - 2y = 57. \end{cases}$

11. $\begin{cases} 40x - 5y = 19, \\ 18x + 65y = 101. \end{cases}$

12. $\begin{cases} \frac{1}{3}(x - 2) = 9 - 3y, \\ 32 = 5x - \frac{1}{4}(5y + 2). \end{cases}$

13. $\begin{cases} 2\frac{1}{3}v + 3\frac{1}{4}y = 74, \\ 4\frac{1}{5}v - 5\frac{1}{6}y = 1. \end{cases}$

14. $\begin{cases} 3\frac{1}{3}s - 215 = -4\frac{1}{2}t, \\ 4\frac{1}{3}t - 3\frac{1}{2}s = 46. \end{cases}$

15. $\begin{cases} \frac{1}{3}Q = 8 - \dfrac{2Q + 3R}{6}, \\ -11 = R - \dfrac{7R - 3Q}{2}. \end{cases}$

16. $\begin{cases} \dfrac{1}{a}(x) + \dfrac{y}{b} = c, \\ ax = -bcy. \end{cases}$

17. $\begin{cases} ax - bz = \frac{1}{2}(a^2 + b^2), \\ (a - b)x = (a + b)z. \end{cases}$

18. $\begin{cases} ax + by = c, \\ a_1x - b_1y = c_1. \end{cases}$

19. $\begin{cases} a'x - b_2y = c', \\ a_3x - b_1y = c_2. \end{cases}$

20. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y - 13 = 0, \\ \frac{1}{5}x + \frac{1}{8}y = 5. \end{cases}$

21. $\begin{cases} \frac{1}{3}u + \frac{1}{4}z - 2 = 0, \\ 3u + 4z - 25 = 0. \end{cases}$

22. $\begin{cases} \frac{1}{7}x + \frac{1}{5}y = \frac{1}{19}, \\ \dfrac{x}{8} - \dfrac{y}{3} = \dfrac{1}{7}. \end{cases}$

23. $\begin{cases} 2\frac{1}{2}x + 7\frac{1}{4}y = 15\frac{3}{4}, \\ \frac{2}{3}x - \frac{5}{7}y = 3. \end{cases}$

24. $\begin{cases} .125x - 2.125y = .3, \\ .375x + .8\frac{1}{3}y = 2\frac{1}{4}. \end{cases}$

25. $\begin{cases} ax - by = P, \\ a'x + b'y = P'. \end{cases}$

26. $\begin{cases} a_2{}^2x + b_3{}^3y = Q^5, \\ a_5{}^3x + ky = A_a. \end{cases}$

27. $\begin{cases} 11\,a'z - 12\,a''S = 3\,A, \\ \frac{2}{3}z - 13\,mS = AB. \end{cases}$

28. $\begin{cases} (a^2-b^2)x+(a^2-b^2)y=55, \\ (a+b)x - (a-b)y = 7. \end{cases}$

339. PROBLEM 2. **To eliminate by substitution.**

Rule. *From the simpler equation find the value of one of the unknown quantities in terms of the other, and substitute this value in the other equation. Then solve in the usual manner.*

Dem. Let (1) $ax + by = c$ and (2) $a'x + b'y = c'$ be a given system. Since the value of x is the same in both equations, the value of x in the first may be substituted for x in the second. Now any solution of (1) and (2) will satisfy (3), *i.e.*, $\frac{a'(c - by)}{a} + b'y = c'$. Hence any solution of the system (1) and (2) will be a solution of (1) and (3). But any solution of (1) and (3) will satisfy (3). Hence any solution of (1) and (3) will satisfy (1) and (2). The system (1) and (3) which is actually solved is equivalent to the system (1) and (2) which was given to be solved.

It is evident that this process eliminates one of the unknown quantities. Show that this is so.

MODEL SOLUTIONS

1. Solve (1) $\frac{f+g}{8} = 5 - \frac{f-g}{6}$ and (2) $\frac{f+g}{4} - \frac{f-g}{3} = 10$.

3. $\quad 7f - g = 120$. Why?
4. $\quad -f + 7g = 120$. Why?
5. $\quad f = 7g - 120$, from (4).
6. $\quad 7(7g - 120) - g = 120$, from (5) and (3).
7. $\quad 48g = 960$.
8. $\quad g = 20$.
9. $\quad f = 140 - 120 = 20$, from (8) and (5).

2. Solve (1) $ax + by = c$ and (2) $a_1x + b_1y = c_1$.

3. $$x = \frac{c - by}{a}, \text{ from (1).}$$

4. $$\frac{a_1c - a_1by}{a} + b_1y = c_1, \text{ substituting (3) in (2).}$$

5. $$a_1c - a_1by + ab_1y = ac_1. \quad \text{Why?}$$

6. $$(ab_1 - a_1b)y = ac_1 - a_1c. \quad \text{Why?}$$

7. $$y = \frac{ac_1 - a_1c}{ab_1 - a_1b}. \quad \text{Why?}$$

Substituting (7) in (1), or better still in this case, finding the value of y instead of x, as in (3), and solving for x, $x = \dfrac{b_1c - bc_1}{ab_1 - a_1b}$.

Let the student work out all the steps.

EXAMPLES

Solve by the method of substitution, and verify:

1. $\begin{cases} 15y + 2x = 10 + 5y, \\ 14x - 5y = 21x - 7. \end{cases}$

2. $\begin{cases} \frac{1}{3}(s - 2) + 3t = 9, \\ 32 = 5s - \frac{1}{4}(5t + 2). \end{cases}$

3. $\begin{cases} x + y - 2l = 0, \\ (l - m)x - (l + m)y = 0. \end{cases}$

4. $\begin{cases} \frac{1}{9}Q + \frac{1}{8}R = 43, \\ \frac{1}{8}Q + \frac{1}{9}R = 42. \end{cases}$

5. $\begin{cases} cv + dz = d^2, \\ v + z = c. \end{cases}$

6. $\begin{cases} \dfrac{a}{b + w} - \dfrac{b}{a - u} = 0, \\ \dfrac{c}{d - u} - \dfrac{d}{c + w} = 0. \end{cases}$

7. $\begin{cases} \frac{2}{7}(15P + 13Q) - (7P - 14Q) = 7, \\ 12(Q - 3 + \frac{2}{3}(Q + \frac{15}{2})) = \frac{2}{7}(15P + 13Q). \end{cases}$

8. $\begin{cases} 2y - \dfrac{z + 3}{4} = \dfrac{1}{5}(3z - 2y) + 7, \\ 4z - \dfrac{8 - y}{3} - 24\frac{1}{2} = -\dfrac{2z + 1}{2}. \end{cases}$

Solve also by Problem 2 Examples 1–8 under Problem 1.

340. PROBLEM 3. **To eliminate by comparison.**

Rule. *Find the value of the same unknown quantity in each equation in terms of the other unknown quantity and known quantities, and place these two values equal to each other. Then solve in the usual manner.*

Dem. Let (1) $a_1x + b_2y = c'$ and (2) $a_2x - b^2y = c_3$ be a given system. Since the value of x is the same in both equations, the value of x obtained from the first may be placed equal to the value of x obtained from the second.

3. $x = \dfrac{c' - b_2y}{a_1}$, from (1).

4. $x = \dfrac{c_3 + b^2y}{a_2}$, from (2).

5. Hence $\dfrac{c' - b_2y}{a_1} = \dfrac{c_3 + b^2y}{a_2}$. Why?

Now any solution of (1) and (2) will satisfy (5). Hence any solution of the system (1) and (2) will be a solution of (1) and (5). But any solution of (1) and (5) will satisfy (5). Hence any solution of (1) and (5) will satisfy (1) and (2). The system (1) and (5) which is actually solved, is equivalent to the system (1) and (2) which was given to be solved. It is evident that this process eliminates one of the unknown quantities.

Questions. 1. What is the value of x in (1) in terms of y, a_1, b_2, and c'? 2. What is the value of x in (2) in terms of y, a_2, b^2, and c_3? 3. Is this 2d system equivalent to the 1st? Why? 4. Will the solution of system (2) satisfy system 1? Why? 5. Is it understood that the answers to questions (1) and (2) are roots of the system? Why? 6. Do (5) $\dfrac{c' - b_2y}{a_1} = \dfrac{c_3 + b^2y}{a_2}$ and (1) $a_1x + b_2y = c'$ form an equivalent system? Will the solution of (5) and (1) satisfy (1) and (2)? 7. From (5) $y = ?$ Substituting this value of y in (1), (5) and (1) are solved, and $x = ?$ and $y = ?$ Are (1) and (2) solved? Are these last values of x and y roots of (1) and (2)? Point out the difference between the first value of x and the last value of x. 8. How may the values of x and y obtained be shown to be the roots of (1) and (2)? 9. Would the surest test be to substitute in (1) or (2), or in both (1) and (2)? Why?

Model Solutions

1. Solve and verify $\begin{cases} (1)\ \dfrac{1}{3x}+\dfrac{y}{5}=1\frac{1}{6}, \\ (2)\ \dfrac{1}{5x}-\dfrac{y}{3}=-\dfrac{47}{30}. \end{cases}$

3. $\dfrac{y}{5}=\dfrac{7}{6}-\dfrac{1}{3x}$, by transposition.

4. $\dfrac{y}{3}=\dfrac{1}{5x}+\dfrac{47}{30}$. Why?

5. $y=\dfrac{35}{6}-\dfrac{5}{3x}$, by multiplying (3) by 5.

6. $y=\dfrac{3}{5x}+\dfrac{47}{10}$, by multiplying (4) by 3.

7. $\dfrac{3}{5x}+\dfrac{47}{10}=\dfrac{35}{6}-\dfrac{5}{3x}$. Why?

8. $\dfrac{3}{5x}+\dfrac{5}{3x}=\dfrac{35}{6}-\dfrac{47}{10}$, by transposition.

9. $\dfrac{34}{15x}=\dfrac{34}{30}$, by collecting terms.

10. $\dfrac{1}{x}=\dfrac{1}{2}$, by reduction.

11. $x=2$, by inversion.

12. $y=\dfrac{35}{6}-\dfrac{5}{6}$, by substituting (11) in (5).

13. $y=5$.

Verification

1. $\begin{cases} \dfrac{1}{6}+\dfrac{5}{5}=1\frac{1}{6}, & \text{by substituting (11) and (13) in (1).} \\ \dfrac{1}{10}-\dfrac{5}{3}=-\dfrac{47}{30}, & \text{by substituting (11) and (13) in (2).} \end{cases}$

2. $\begin{cases} 1\frac{1}{6}\equiv 1\frac{1}{6} & \text{by collecting terms.} \\ -\dfrac{47}{30}\equiv-\dfrac{47}{30}, & \text{by collecting terms.} \end{cases}$

2. Solve and verify $\begin{cases} (1)\ ax_1 + by' = c^2, \\ (2)\ a'x_1 + b_1y' = c_2. \end{cases}$

3. $$x_1 = \frac{c^2 - by'}{a}.$$

4. $$x_1 = \frac{c_2 - b_1y'}{a'}.$$

5. $$\frac{c^2 - by'}{a} = \frac{c_2 - b_1y'}{a'}.$$

6. $a'c^2 - a'by' = ac_2 - ab_1y'.$

7. $(ab_1 - a'b)y' = ac_2 - a'c^2.$

8. $\therefore \quad y' = \dfrac{ac_2 - a'c^2}{ab_1 - a'b}.$

9. from (1), $y' = \dfrac{c^2 - ax_1}{b}.$

10. from (2), $y' = \dfrac{c_2 - a'x_1}{b_1}.$

11. $$\frac{c^2 - ax_1}{b} = \frac{c_2 - a'x_1}{b_1}.$$

12. $b_1c^2 - ab_1x_1 = bc_2 - a'bx_1.$

13. $(a'b - ab_1)x_1 = bc_2 - b_1c^2.$

14. $$x_1 = \frac{bc_2 - b_1c^2}{a'b - ab_1}, \text{ or } \frac{b_1c^2 - bc_2}{ab_1 - a'b}.$$

Verification

1. $$\begin{cases} -\dfrac{abc_2 - ab_1c^2}{ab_1 - a'b} + \dfrac{abc_2 - a'bc^2}{ab_1 - a'b} = c^2, \\ -\dfrac{a'bc_2 - a'b_1c^2}{ab_1 - a'b} + \dfrac{ab_1c_2 - a'b_1c^2}{ab_1 - a'b} = c_2. \end{cases}$$

2. $$\begin{cases} \dfrac{(ab_1 - a'b)c^2}{ab_1 - a'b} = c^2, \\ \dfrac{(ab_1 - a'b)c_2}{ab_1 - a'b} = c_2. \end{cases}$$

3. $$\begin{cases} c^2 \equiv c^2, \\ c_2 \equiv c_2. \end{cases}$$

Verify these roots by Suggestion 13, page 221.

EXAMPLES

Solve by the method of comparison and verify:

1. $\begin{cases} 8y - 2x = 36, \\ 6x - 158 = -14y. \end{cases}$

2. $\begin{cases} 2x - \frac{5}{2}y = 0, \\ 5 = 1\frac{1}{2}x - \frac{5}{4}y. \end{cases}$

3. $\begin{cases} \dfrac{14}{y} - \dfrac{x}{20} = 32, \\ \dfrac{x}{12} + \dfrac{16}{y} = 45. \end{cases}$

4. $\begin{cases} (a+b)y - (a-b)z - 4ab = 0, \\ (a-b)y - (a+b)z - 0 = 0. \end{cases}$

5. $\begin{cases} 3(p-q) + 7(p+q) - 80 = 0, \\ 7(p+q) - 3(p-q) - 32 = 0. \end{cases}$

6. $\begin{cases} (a^2 - b^2)(x + y) = 2a^2 + 2b^2, \\ (a^2 - b^2)(x - y) = 4ab. \end{cases}$

7. $\begin{cases} \dfrac{x-y}{x+y} = 3 - \dfrac{10}{x+y}, \\ \dfrac{\dfrac{9x+8}{6}}{\dfrac{2x+3y}{6} - \dfrac{3x-4y}{5}} - 11\frac{2}{3} = 0. \end{cases}$

8. $\begin{cases} \frac{1}{5}(x + 2y) - \frac{1}{6}(3x + 2y) + \frac{4}{15} = \frac{1}{10}y, \\ \frac{1}{8}(3x - 5y) + \frac{1}{6}(4x + 5) - 1 = \frac{1}{4}(11x + 5y). \end{cases}$

Solve also by Problem 3 Examples 1–16 under Problem 2.

341. PROBLEM 4. **To eliminate by arbitrary multipliers.**

Rule — 1. *Multiply both members of one of the given equations, simplified if necessary, by some arbitrary quantity as* k, *and to the resulting equation add the other given equation, member by member.*

2. *Give to the arbitrary quantity two values in succession which will render,* (1) *the coefficient of one unknown quantity zero, and then* (2) *the coefficient of the other unknown quantity zero.*

Dem. The first operation is performed by Axiom 1. (State this Axiom.) The second operation is legitimate, for, since the equation is true for all values of k (except zero), any number may be substituted for k.

Model Solutions

1. Solve $\begin{cases} (1)\ 6x + 5y = 128, \\ (2)\ 3x + 4y = 88. \end{cases}$

3. $\begin{cases} 6x + 5y = 128, \\ 3kx + 4ky = 88k, \text{ by multiplying (2) by } k. \end{cases}$
4.
5. $(6 + 3k)x + (5 + 4k)y = 128 + 88k.$

To make the coefficient of x zero, $3k+6$ must be put equal to 0. If $3k + 6 = 0$, $k = -2$.

6. $\therefore\ 0x + (5 - 8)y = 128 - 176.$
7. $-3y = -48.$
8. $y = 16.$

That the coefficient of y shall become zero, $4k + 5$ must be put equal to 0. If $4k + 5 = 0$, then $k = -\frac{5}{4}$.

9. $\therefore\ (6 - \frac{15}{4})x + 0y = 128 - 110.$
10. $\frac{9}{4}x = 18.$
11. $x = 8.$

2. Solve $\begin{cases} (1)\ ax + by = c, \\ (2)\ a'x + b'y = c'. \end{cases}$

3. $\begin{cases} akx + bky = ck, \text{ by multiplying (1) by } k. \\ a'x + b'y = c'. \quad (2) \end{cases}$
4.
5. $(ak + a')x + (bk + b')y = ck + c'.$

Put $ak + a' = 0$; then $k = -\frac{a'}{a}$.

6. $\therefore\ 0x + \left(-\frac{a'b}{a} + b'\right)y = -\frac{a'c}{a} + c'.$
7. $y = \frac{ac' - a'c}{ab' - a'b}.$

Put $bk + b' = 0$; then $k = -\frac{b'}{b}$.

8. $\therefore\ \left(-\frac{ab'}{b} + a'\right)x + 0y = -\frac{b'c}{b} + c'.$
9. $x = \frac{bc' - b'c}{a'b - ab'}$, or $\frac{b'c - bc'}{ab' - a'b}.$

Question. By the principle of symmetry, how may the value of x be written immediately from the value of y?

EXAMPLES

Solve by the method of arbitrary multipliers:

1. $\begin{cases} 3x + 2y = 26, \\ 5x - 2y = 38. \end{cases}$

2. $\begin{cases} x + 4y - 102 = 0, \\ 4x + y - 48 = 0. \end{cases}$

3. $\begin{cases} 3y - 2x = -1, \\ 5y = 4x - 4. \end{cases}$

4. $\begin{cases} ax + 2by - c = 0, \\ ax + 3a^2 - b^2 = y. \end{cases}$

5. $\begin{cases} \frac{1}{3}x - \frac{1}{4}y = 2, \\ \frac{1}{4}x - \frac{1}{2}y = 7. \end{cases}$

6. $\begin{cases} mx - 3x - y = a, \\ -3x + 2amy + n = 0. \end{cases}$

7. $\begin{cases} \dfrac{x+1}{y} = \dfrac{1}{3}, \\ \dfrac{x}{y+4} = \dfrac{1}{4}. \end{cases}$

8. $\begin{cases} a'x' + b_1y_1 = c_2, \\ a_1x' - b'y_1 = c^2. \end{cases}$

9. $\begin{cases} ax - by + c = 0, \\ cx + dy - e = 0. \end{cases}$

10. $\begin{cases} x - \dfrac{2y-x}{23-x} = 20 - \dfrac{59-2x}{2}, \\ y + \dfrac{y-3}{x-18} = 30 - \dfrac{73-3y}{3}. \end{cases}$

342. PROBLEM 5. **To eliminate by the rule of cross-multiplication.**

Rule — 1. *Write the coefficients of x and of y, and the unknown term, in order under one another, beginning and ending with the coefficients of y.*

$$1.\quad ax + by - c = 0.$$

$$2.\quad a'x + b'y - c' = 0.$$

$$\begin{array}{cccc} b & -c & a & b \\ b' & -c' & a' & b' \end{array}$$

2. *From the product of each upper left-hand quantity and its diagonal, subtract the product of the lower left-hand quantity and its diagonal. The three results obtained are in proportion to the two unknown quantities and to unity.*

Hence

$$\frac{x}{-bc' + b'c} = \frac{y}{-ca' + c'a} = \frac{1}{ab' - a'b}.$$

Dem. That these results are true may be shown from the results obtained under Problem 1, model solution No. 2. The values of x and y there obtained from the equations

$$ax + by = c$$

and $$a'x + b'y = c' \text{ were}$$

1. $$x = \frac{b'c - bc'}{ab' - a'b}.$$

2. $$y = \frac{ac' - a'c}{ab' - a'b}.$$

Dividing both members of (1) by $b'c - bc'$ and both members of (2) by $ac' - a'c$,

3. $$\frac{x}{b'c - bc'} = \frac{1}{ab' - a'b}.$$

4. $$\frac{y}{ac' - a'c} = \frac{1}{ab' - a'b}.$$

$$\therefore \frac{x}{b'c - bc'} = \frac{y}{ac' - a'c} = \frac{1}{ab' - a'b}.$$

343. When these cross-product differences are written in the form of square matrices, they are called **Determinants**.

Thus $$\begin{vmatrix} b & -c \\ b' & -c' \end{vmatrix}, \quad \begin{vmatrix} -c & a \\ -c' & a' \end{vmatrix}, \quad \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}.$$

Note. Students preparing for college should become familiar with this last method. The teacher may use and extend it according to the demands of the school.

Model Solutions

1. Solve $\begin{cases} 3x + 3 = y, \\ 4x = y + 1. \end{cases}$

$$\begin{cases} 3x - y + 3 = 0. \\ 4x - y - 1 = 0. \end{cases}$$

$$\begin{array}{cccccccc} -1 & & 3 & & 3 & & -1 \\ & \times & & \times & & \times & \\ -1 & & -1 & & 4 & & -1 \end{array}$$

$$\frac{x}{1+3} = \frac{y}{12+3} = \frac{1}{-3+4}.$$

$$\frac{x}{4} = \frac{1}{1}, \text{ or } x = 4.$$

$$\frac{y}{15} = \frac{1}{1}, \text{ or } y = 15.$$

2. Solve $\begin{cases} ax - by + c = 0, \\ a'x - b'y + c' = 0. \end{cases}$

$$\begin{vmatrix} -b & c \\ -b' & c' \end{vmatrix}, \quad \begin{vmatrix} c & a \\ c' & a' \end{vmatrix}, \quad \begin{vmatrix} a & -b \\ a' & -b' \end{vmatrix}.$$

$$\frac{x}{-bc' + b'c} = \frac{y}{ca' - c'a} = \frac{1}{-ab' + a'b}.$$

$$x = \frac{bc' - b'c}{ab' - a'b}.$$

$$y = \frac{c'a - ca'}{ab' - a'b}.$$

3. Solve $\begin{cases} 5x + 2y + 5 = 0, \\ -3x - 4y + 2 = 0. \end{cases}$

$$x = \frac{\begin{vmatrix} 2 & 5 \\ -4 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ -3 & -4 \end{vmatrix}} = \frac{24}{-14} = -\frac{12}{7}. \qquad y = \frac{\begin{vmatrix} 5 & 5 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ -3 & -4 \end{vmatrix}} = \frac{-25}{-14} = \frac{25}{14}.$$

EXAMPLES

Solve by the rule of cross-multiplication and verify:

1. $\begin{cases} a'x + b'y + c' = 0, \\ ax + by + c = 0. \end{cases}$

2. $\begin{cases} 5x - 4 = 3y, \\ 10 + 7x - 12y = 0. \end{cases}$

3. $\begin{cases} \frac{1}{4}x - \frac{1}{2}y - 1 = 0, \\ \frac{1}{12}x + \frac{1}{6}y = 1. \end{cases}$

4. $\begin{cases} 6x + 4y = 24, \\ 3x + 15y = 51. \end{cases}$

5. $\begin{cases} mx - my + m = x + y - 1, \\ mnx - mny + mn = 1 - x + y. \end{cases}$

6. $\begin{cases} 2x - 7y = 41, \\ 3x - 4y = 42. \end{cases}$

7. $\begin{cases} 5y - 2x - 21 = 0, \\ 13x - 120 = 4y. \end{cases}$

8. $\begin{cases} abx + cdy = 2, \\ abdx - bcdy = d - b. \end{cases}$

9. $\begin{cases} \dfrac{x-2}{5} - \dfrac{10-x}{3} = \dfrac{y-10}{4}, \\ \frac{1}{8}(2y+4) = \frac{1}{8}(4x+y+13). \end{cases}$

10. $\begin{cases} (m^3 + n^3)x = (m^3 - n^3)y, \\ (y - m)(m - n) = (m + n)(x - m). \end{cases}$

SECTION II

SYSTEMS OF FRACTIONAL EQUATIONS

344. When the system of equations involves **the reciprocals of the unknown quantities**, it is generally better to solve without clearing of fractions during the process of elimination. The reason for this is obvious when it is remembered that the solution is made complicated by the introduction of new roots.

MODEL SOLUTIONS

1. Solve $\begin{cases} (1)\ \dfrac{3}{2x} - \dfrac{5}{3y} = \dfrac{5}{12}, \\ (2)\ \dfrac{1}{3x} + \dfrac{2}{7y} = \dfrac{3}{21}. \end{cases}$

3. $\begin{cases} \dfrac{9}{x} - \dfrac{10}{y} = \dfrac{5}{2}, & \text{by multiplying (1) by 6.} \\ \dfrac{7}{x} + \dfrac{6}{y} = 3, & \text{by multiplying (2) by 21.} \end{cases}$
4.

5. $\begin{cases} \dfrac{27}{x} - \dfrac{30}{y} = \dfrac{15}{2}, & \text{by multiplying (3) by 3.} \\ \dfrac{35}{x} + \dfrac{30}{y} = 15, & \text{by multiplying (4) by 5.} \end{cases}$
6.

7. $\dfrac{62}{x} = \dfrac{45}{2}$, by adding (5) and (6).

8. $\dfrac{x}{62} = \dfrac{2}{45}$, by inversion.

9. $x = \frac{124}{45}$, by multiplying (8) by 62.

10. $\begin{cases} \dfrac{63}{x} - \dfrac{70}{y} = \dfrac{35}{2}, & \text{by multiplying (3) by 7.} \\ \dfrac{63}{x} + \dfrac{54}{y} = \dfrac{54}{2}, & \text{by multiplying (4) by 9.} \end{cases}$
11.

12. $\dfrac{124}{y} = \dfrac{19}{2}$, by subtracting (10) from (11).

13. $\dfrac{y}{124} = \dfrac{2}{19}$, by inversion.

14. $y = \frac{248}{19}$, by multiplying (13) by 124.

Verification

1. $\dfrac{3 \times 45}{248} - \dfrac{5 \times 19}{3 \times 248} = \dfrac{5}{12}$, $\qquad \dfrac{15}{124} + \dfrac{19}{7 \times 124} = \dfrac{1}{7}$.

2. $\dfrac{3 \times 9}{62} - \dfrac{19}{3 \times 62} = \dfrac{1}{3}$, $\qquad \dfrac{124}{7 \times 124} = \dfrac{1}{7}$.

3. $\frac{1}{3} \equiv \frac{1}{3}$, $\qquad \frac{1}{7} \equiv \frac{1}{7}$.

2. Solve $\begin{cases} (1) \ 3y - 2x = 5xy. \\ (2) \ 5x + 4y = xy. \end{cases}$

$$\begin{aligned}
&3. \quad \frac{3}{x} - \frac{2}{y} = 5, && \text{by dividing (1) by } xy. \\
&4. \quad \frac{4}{x} + \frac{5}{y} = 1, && \text{by dividing (2) by } xy. \\
&5. \quad \frac{15}{x} - \frac{10}{y} = 25, && \text{by multiplying (3) by 5.} \\
&6. \quad \frac{8}{x} + \frac{10}{y} = 2, && \text{by multiplying (4) by 2.} \\
&7. \quad \frac{23}{x} = 27, && \text{by adding (5) and (6).} \\
&8. \quad x = \tfrac{23}{27}. && \\
&\quad\;\; y = (?) &&
\end{aligned}$$

EXAMPLES

Solve without clearing of fractions:

1. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c, \\ \dfrac{b}{x} + \dfrac{c}{y} = d. \end{cases}$

2. $\begin{cases} \dfrac{3}{x} - \dfrac{7}{y} = \dfrac{5}{3}, \\ \dfrac{2}{x} - \dfrac{5}{y} = \dfrac{7}{5}. \end{cases}$

3. $\begin{cases} \dfrac{7}{w} + \dfrac{4}{z} = 1\tfrac{2}{3}, \\ \dfrac{1}{4} - \dfrac{2}{z} = -\dfrac{3}{w}. \end{cases}$

4. $\begin{cases} \dfrac{a}{l} + \dfrac{b}{m} = 1, \\ \dfrac{b}{l} + \dfrac{a}{m} = 1. \end{cases}$

5. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = s, \\ \dfrac{c}{x} - \dfrac{d}{y} = t. \end{cases}$

6. $\begin{cases} \dfrac{4}{y} - 1 = \dfrac{4}{x}, \\ \dfrac{2}{x} - \dfrac{4}{y} = -\dfrac{3}{2}. \end{cases}$

7. $\begin{cases} \frac{1}{x} - \frac{1}{y} = k, \\ \frac{1}{x} + \frac{1}{y} = g. \end{cases}$

8. $\begin{cases} \frac{12}{x} + \frac{12}{y} = 1, \\ \frac{9}{x} + \frac{6}{y} = \frac{7}{10}. \end{cases}$

9. $\begin{cases} \frac{2}{ax} + \frac{3}{by} = 5, \\ \frac{5}{ax} - \frac{3}{by} = 2. \end{cases}$

10. $\begin{cases} 7y + 5x = 19xy, \\ 8y - 3x = 7xy. \end{cases}$

11. $\begin{cases} 5x - 2y = 4xy, \\ 4x - 3y = 9xy. \end{cases}$

Solve by any method and verify:

1. $\begin{cases} (a+b)x + (a-b)y = a^2 - b^2, \\ (a-b)x + (a+b)y = a^2 + b^2. \end{cases}$

2. $\begin{cases} \frac{1}{c}x + \frac{1}{d}y = cd, \\ x + y = cd(c+d). \end{cases}$

3. $\begin{cases} 7ax - 2by = c, \\ 2ax - 7cy = d. \end{cases}$

4. $\begin{cases} ax + by = 2ab, \\ bx - ay = b^2 - a^2. \end{cases}$

5. $\begin{cases} \frac{3}{x} - \frac{5}{y} = -2, \\ \frac{5}{x} - \frac{3}{y} = 2. \end{cases}$

6. $\begin{cases} \frac{1}{2}(x-y) + \frac{3y}{8} = \frac{7}{8}, \\ \frac{1}{5}(3x+y) - \frac{4}{15}x = \frac{1}{10}(5x+y). \end{cases}$

7. $\begin{cases} 7x - 3y - 10 = 0, \\ 3x - \frac{2y-5}{x-2} + \frac{4-9x}{3} = 0. \end{cases}$

8. $\begin{cases} \frac{3}{x} + \frac{1}{y} = 1\frac{1}{4}, \\ \frac{2}{x} - \frac{3}{y} = -1. \end{cases}$

9. $\begin{cases} \frac{x}{a+b} - \frac{y}{a-b} = \frac{1}{a+b}, \\ \frac{x}{a+b} + \frac{y}{a-b} = \frac{1}{a-b}. \end{cases}$

10. $\begin{cases} \frac{1}{8}(x-1) + \frac{1}{5}(y-2) = 2, \\ 2x + \frac{1}{8}(2y-5) = 21. \end{cases}$

11. $\begin{cases} x + y = a + b, \\ (a+c)x - by - bc = 0. \end{cases}$

12. $\begin{cases} \frac{b}{x} + \frac{a}{y} - a^2 = b^2, \\ \frac{a}{bx} + \frac{b}{ay} - a - b = 0. \end{cases}$

13. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=1, \\ \dfrac{a}{x}+\dfrac{b}{y}=4. \end{cases}$

14. $\begin{cases} \dfrac{m}{x}+\dfrac{n}{x_1}=a, \\ \dfrac{n}{x}+\dfrac{m}{x_1}=a_1. \end{cases}$

15. $\begin{cases} lx+my=n, \\ gx+hy=k. \end{cases}$

16. $\begin{cases} a_1x'+b_1y'=c_1, \\ a'x'-b'y'=c'. \end{cases}$

17. $\begin{cases} \frac{1}{2}x'+\frac{1}{5}y'=5, \\ \dfrac{2}{x'}+\dfrac{5}{y'}=\dfrac{5}{6}. \end{cases}$

18. $\begin{cases} \dfrac{1}{3x_1}+\dfrac{1}{5x_2}=\dfrac{2}{9}, \\ \dfrac{1}{5x_1}+\dfrac{1}{3x_2}=\dfrac{1}{4}. \end{cases}$

19. $\begin{cases} s+t=c, \\ (a-b)s=(a+b)t. \end{cases}$

20. $\begin{cases} 2.5\,P+.4\,Q-8=0, \\ .6\,P-.1\,Q-.695=0. \end{cases}$

21. $\begin{cases} (v+z)^2-v^2-6\,v=z^2, \\ (1+v)^2+(1+z)^2-v^2-z^2=12. \end{cases}$

22. $\begin{cases} aw-bu=\dfrac{a^2+b^2}{a^2-b^2}, \\ 2(a-b)w+(3\,a+3\,b)u-5=0. \end{cases}$

23. $\begin{cases} \dfrac{\frac{1}{3}(x+y)-\frac{1}{5}(2\,x+3\,y-1)}{5\,x-2\,y+7}+\dfrac{1}{9}=0, \\ 7\,ax-3\,ay-4\,a^2+6\,a=x+6-5\,y. \end{cases}$

24. $\begin{cases} (2\,x-2\,y):(3\,y-5\,x)::(5\,y):(3\,x-6\,y), \\ (2\,x-5):(2\,y)::(x-3):(y+1). \end{cases}$

PROBLEMS LEADING TO SYSTEMS OF EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES

1. The first digit of a certain number exceeds the second digit by 4; and when the number is divided by the sum of its digits the quotient is 7. What is the number?

2. Find two numbers whose sum and difference are as $4:1$ and whose sum and product as $4:15$.

3. The smaller of two numbers divided by the greater $= \frac{2}{7}$, and the greater divided by the less gives a quotient of 3 and a remainder of 1. Find the numbers.

4. I invested \$5800 in 5% stock at 105 and 3% stock at 96. How much did I invest in each kind if my annual income is \$250?

5. The combined wages of 6 carpenters and 2 painters for one day is \$28. If for another day's work the same sum is paid to 5 carpenters and 4 painters, what is the daily wages of each?

6. If a father was 4 times as old as his son 7 years ago, and will be twice as old as his son 7 years from now, what is the present age of each?

7. The sum of two fractions whose numerators are 3, is 3 times the smaller; and 3 times the smaller subtracted from twice the greater gives $\frac{3}{8}$. What are the fractions?

8. A gives B 100 yards' start in a race, gains 750 feet on him in the first 9000 feet, and overtakes him in 4 minutes. Find the rate of each.

9. A man can row 8 miles down stream in 40 minutes, and 14 miles up stream in 1 hour and 45 minutes. Find the rate of the current and the rate the man could row in still water.

10. A and B have 45 coins between them which are in dollars and dimes. A has 4 times as many dollars as dimes, and B has just as many dollars as dimes. If A has \$9.50 more than B, how much money has each?

11. A train 72 yards long passed another train 60 yards long, which was going in the opposite direction, in 4 seconds. Had the first-mentioned train been traveling at twice its actual speed, the trains would have passed each other in 3 seconds. Find the number of miles an hour at which the trains were traveling.

12. A number consists of two digits. It is 3 more than 4 times the sum of its digits, and when subtracted from 121 the digits are found to have changed places. What is the number?

13. If a given rectangle had been 3 feet longer and 2 feet wider, it would have contained 64 square feet more. But if it had been 2 feet longer and 3 feet wider, it would have contained 68 square feet more. What are the dimensions of the rectangle?

14. A grocer mixed tea that cost him 42 cents a pound with tea that cost him 54 cents a pound. He had 30 pounds of the mixture, and by selling it at the rate of 60 cents a pound he gained as much as 10 pounds of the cheaper tea cost him. How many pounds of each did he put in the mixture?

15. A person rows down a stream, which flows at the rate of 4 miles an hour, for a certain distance in 1 hour and 40 minutes. In returning it takes him 4 hours and 15 minutes to arrive at a point 3 miles short of his starting place. Find the distance he pulled down the stream and the rate of his rowing.

16. A servant is given \$ 9 with which to buy 4 kilograms of tea and 7 kilograms of coffee, and told to expect 15 cents in change; but she makes a mistake and buys 7 kilograms of tea and 4 kilograms of coffee and receives a bill of 30 cents due. What is the price of the tea a kilogram and of the coffee a pound? (1 kilogram = 2.2 lb.)

17. Gold weighs $19\frac{1}{4}$ times as much as water; silver $10\frac{1}{2}$ times. One cubic foot of water weighs $62\frac{1}{2}$ lb. A goldsmith offers a mass of $\frac{1}{4}$ cu. ft., which he asserts to be gold, weighing $257\frac{13}{16}$ lb. If alloyed, what is the ratio of silver to gold (*a*) by weight? (*b*) by volume?

18. A and B walk around a circle whose circumference is C. If they start from the same point and go in opposite directions, they meet in 4 hours. If they start from opposite points and walk around the circle in the same direction, they meet in 8 hours. Find each one's rate.

SECTION III

THREE OR MORE UNKNOWN QUANTITIES

345. PROBLEM. **To solve a system of linear equations involving three or more unknown quantities.**

Rule — 1. *Decide upon the unknown quantity first to be eliminated, say z. Eliminate z from all the equations by addition or subtraction, using two equations at a time, and obtain a new system containing as many equations as unknown quantities, in which z does not appear.*

2\. *Repeat this process of elimination on each successive system until but one equation of one unknown quantity remains.*

3\. *Find the value of the remaining unknown quantity and substitute this value in a preceding equation containing this unknown quantity and one other, thus finding the value of a second unknown quantity. Continue to substitute backwards until the values of all the unknown quantities have been found, when the system is said to be solved.*

MODEL SOLUTIONS

1\. Solve $\begin{cases} (1)\quad x+y+z=15, \\ (2)\quad 4x+2y+3z=53, \\ (3)\quad 5x+3y-4z=44. \end{cases}$

$$
\begin{array}{ll}
4. & 3x+3y+3z=45, \text{ by multiplying (1) by 3.} \\
5. & \underline{4x+2y+3z=53} \quad (2) \\
6. & x-y = 8, \text{ by subtracting (4) from (5).} \\
7. & 4x+4y+4z=60, \text{ by multiplying (1) by 4.} \\
8. & \underline{5x+3y-4z=44} \quad (3) \\
9. & 9x+7y = 104, \text{ by adding (7) and (8).} \\
10. & 7x-7y=56, \text{ by multiplying (6) by 7.} \\
11. & \underline{9x+7y=104} \quad (9) \\
12. & 16x = 160. \quad \therefore x=10. \\
13. & y=2, \text{ from (6) and (12).} \\
14. & z=3, \text{ from (1), (12), and (13).}
\end{array}
$$

2. Solve $\begin{cases} (1) & 3u + x + 2y - z = 22, \\ (2) & 4x - y + 3z = 35, \\ (3) & 4u + 3x - 2y = 19, \\ (4) & 2u + 4y + 2z = 46. \end{cases}$

Suppose it is decided to eliminate u first.

$$\begin{array}{lrl} 5. & 12u + 4x + 8y - 4z = 88, & \text{by multiplying (1) by 4.} \\ 6. & 12u + 9x - 6y \qquad = 57, & \text{by multiplying (3) by 3.} \\ \hline 7. & -5x + 14y - 4z = 31, & \text{by subtracting (6) from (5).} \end{array}$$

$$\begin{array}{lrl} 8. & 4u + 3x - 2y \qquad = 19 & (3) \\ 9. & 4u + \qquad 8y + 4z = 92, & \text{by multiplying (4) by 2.} \\ \hline 10. & 3x - 10y - 4z = -73, & \text{by subtracting (9) from (8).} \end{array}$$

The first new system is

$$\begin{array}{ll} 2. & \\ 7. & \\ 10. & \end{array} \begin{cases} 4x - y + 3z = 35 \\ -5x + 14y - 4z = 31 \\ 3x - 10y - 4z = -73 \end{cases}$$

Suppose it is decided to eliminate z next.

$$\begin{array}{lrl} 11. & 16x - 4y + 12z = 140, & \text{by multiplying (2) by 4.} \\ 12. & -15x + 42y - 12z = 93, & \text{by multiplying (7) by 3.} \\ \hline 13. \text{ Second new} & x + 38y = 233, & \text{by adding (11) and (12).} \\ 14. \quad \text{system.} & -x + 3y = 13, & \text{by subtracting (10) from (7) and dividing by 8.} \\ \hline 15. & 41y = 246, & \text{by adding (13) and (14).} \\ 16. & y = 6 & \\ 17. & x = 5, & \text{from (14) and (16).} \\ 18. & z = 7, & \text{from (2), (16), and (17).} \\ 19. & u = 4, & \text{from (4), (16), and (18).} \end{array}$$

3. Solve $\begin{cases} (1) & \frac{2}{3}x + \frac{3}{5}y + \frac{1}{2}z = 17, \\ (2) & \frac{5}{6}x + \frac{4}{5}y + \frac{1}{4}z = 18, \\ (3) & \frac{3}{4}x + \frac{2}{3}y + \frac{1}{3}z = 19. \end{cases}$

4. $\frac{5}{3}x + \frac{8}{5}y + \frac{1}{2}z = 36$, by multiplying (2) by 2.

5. $\frac{2}{3}x + \frac{3}{5}y + \frac{1}{2}z = 17$, (1)

6. $x + y = 19$, by subtracting (5) from (4).

7. $\frac{2}{9}x + \frac{1}{5}y + \frac{1}{6}z = \frac{17}{3}$, by dividing (1) by 3.

8. $\frac{3}{8}x + \frac{1}{3}y + \frac{1}{6}z = \frac{19}{2}$, by dividing (3) by 2.

9. $\frac{11}{72}x + \frac{2}{15}y = \frac{23}{6}$, by subtracting (7) from (8).

10. $\frac{11}{24}x + \frac{2}{5}y = \frac{23}{2}$, by multiplying (9) by 3.

11. $\frac{2}{5}x + \frac{2}{5}y = \frac{38}{5}$, by multiplying (6) by $\frac{2}{5}$.

12. $\frac{7}{120}x = \frac{39}{10}$, by subtracting (11) from (10).

13. $x = \frac{468}{7}$.

14. $y = -\frac{335}{7}$, from (13) and (6).

15. $z = \frac{16}{7}$, from (13), (14), and (1).

4\. Solve

$$\begin{cases} (1) & \dfrac{3}{x} - \dfrac{4}{5y} + \dfrac{1}{z} = 7\frac{3}{5}, \\ (2) & \dfrac{1}{3x} + \dfrac{1}{2y} + \dfrac{2}{z} = 10\frac{1}{6}, \\ (3) & \dfrac{4}{5x} - \dfrac{1}{2y} + \dfrac{4}{z} = 16\frac{1}{10}. \end{cases}$$

4. $\frac{15}{x} - \frac{4}{y} + \frac{5}{z} = 38$, by multiplying (1) by 5.

5. $\frac{2}{x} + \frac{3}{y} + \frac{12}{z} = 61$, by multiplying (2) by 6.

6. $\frac{8}{x} - \frac{5}{y} + \frac{40}{z} = 161$, by multiplying (3) by 10.

7. $\frac{8}{x} + \frac{12}{y} + \frac{48}{z} = 244$, by multiplying (5) by 4.

8. $\frac{8}{x} - \frac{5}{y} + \frac{40}{z} = 161$ (6).

9. $\frac{17}{y} + \frac{8}{z} = 83$, by subtracting (8) from (7).

10. $\frac{30}{x} - \frac{8}{y} + \frac{10}{z} = 76$, by multiplying (4) by 2.

11. $\frac{30}{x} + \frac{45}{y} + \frac{180}{z} = 915$, by multiplying (5) by 15.

12. $\frac{53}{y} + \frac{170}{z} = 839$, by subtracting (10) from (11).

13. $\frac{1445}{y} + \frac{680}{z} = 7055$, by multiplying (9) by 85.

14. $\frac{212}{y} + \frac{680}{z} = 3356$, by multiplying (12) by 4.

15. $\frac{1233}{y} = 3699$, by subtracting (14) from (13).

16. $\frac{y}{1233} = \frac{1}{3699}$, by inversion.

17. $y = \frac{1}{3}$.

18. $z = \frac{1}{4}$, from (17) and (7).

19. $x = \frac{1}{2}$, from (5), (17), and (18).

5. Solve $\begin{cases} (1)\ x + y = a, \\ (2)\ y + z = b, \\ (3)\ x + z = c. \end{cases}$

4. $x + y = a$ (1).
5. $y + z = b$ (2).
6. $x - z = a - b$, by subtracting (5) from (4).
7. $x + z = c$ (3).
8. $2x = a - b + c$, by adding (6) and (7).
9. $2z = -a + b + c$, by subtracting (6) from (7).
10. $x = \frac{1}{2}(a - b + c)$.
11. $z = \frac{1}{2}(-a + b + c)$.
12. $y = \frac{1}{2}(a + b - c)$, by symmetry.

Explain how (12) was obtained by symmetry.

EXAMPLES

Solve and verify :

1. $\begin{cases} x+2y+3z=17, \\ 2x-3y+z=0, \\ 3x+y-5z=-15. \end{cases}$

2. $\begin{cases} 5x+3y+0z=65, \\ 0x+2y-z=11, \\ 3x+0y+4z=57. \end{cases}$

3. $\begin{cases} 3x-5y+4z=5, \\ 7x+2y-3z=2, \\ 4x+3y-z=7. \end{cases}$

4. $\begin{cases} 8s-t+6u=64, \\ s-2t+3u=17, \\ 3s+2t-2u=7. \end{cases}$

5. $\begin{cases} 5x+3y+3z=48, \\ 2x+6y-3z=18, \\ 8x-3y+2z=21. \end{cases}$

6. $\begin{cases} 4y-3z=1, \\ 5x-2z=2, \\ 3x-2y=8. \end{cases}$

7. $\begin{cases} 3x-5y+4z=5, \\ 7x+2y-3z=2, \\ 4x+3y-z=7. \end{cases}$

8. $\begin{cases} x+y=2c, \\ x+z=2b, \\ y+z=2a. \end{cases}$

9. $\begin{cases} 2x+3y+4z=16, \\ x+2y-5z=2, \\ 5x-6y+3z=6. \end{cases}$

10. $\begin{cases} ax+by=c, \\ dx+ez=f, \\ gy+hz=i. \end{cases}$

11. $\begin{cases} 2x+3y+4z=29, \\ 3x+2y+5z=32, \\ 4x+3y+2z=25. \end{cases}$

12. $\begin{cases} 9x-10y-33z=4, \\ 7x-3y-15z=3, \\ 2x+4y+27z=28. \end{cases}$

13. $\begin{cases} 10x+15y=24z+41, \\ 15x=12y-16z+10, \\ 18x-7z=14y-13. \end{cases}$

14. $\begin{cases} \frac{1}{2}x+\frac{2}{3}y+\frac{3}{5}z=17, \\ \frac{1}{3}x+\frac{3}{4}y+\frac{2}{3}z=19, \\ \frac{1}{4}x+\frac{5}{6}y+\frac{4}{5}z=18. \end{cases}$

15. $\begin{cases} \dfrac{v+w}{2}-\dfrac{z-w}{4}+w=4\frac{1}{2}, \\ \dfrac{2w-3}{3}+\dfrac{w-v}{2}+z=7, \\ \dfrac{6w+v}{6}-\dfrac{4z-6w}{4}+v=3\frac{2}{3}. \end{cases}$

16. $\begin{cases} x+\dfrac{y}{2}+\dfrac{z}{3}-6=0, \\ y+\dfrac{z}{2}+\dfrac{x}{3}+1=0, \\ z+\dfrac{x}{2}+\dfrac{y}{3}=17. \end{cases}$

17. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}=a, \\ \dfrac{1}{x}-\dfrac{1}{y}+\dfrac{1}{z}=b, \\ -\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=c. \end{cases}$

18. $\begin{cases} x+y=5xy, \\ y+z=6yz, \\ \dfrac{xz}{x+z}=\dfrac{1}{7}. \end{cases}$

19. $\begin{cases} 2yz+xz=3xy, \\ \dfrac{1}{x}+\dfrac{0}{y}+\dfrac{1}{z}=\dfrac{4}{3}, \\ \dfrac{0}{x}-\dfrac{2}{y}+\dfrac{3}{z}=2. \end{cases}$

20. $\begin{cases} ax+by=l, \\ cy+dz=m, \\ ex+fz=n. \end{cases}$

21. $\begin{cases} 2x+3y=39, \\ 4y+3z=41, \\ 3u-2y=2, \\ 5x-7z=11. \end{cases}$

22. $\begin{cases} x+y+z=6, \\ x+y+u=7, \\ x+z+u=8, \\ y+z+u=9. \end{cases}$

23. $\begin{cases} 2x+3y+4z=5, \\ 6x+7y+8z=9, \\ 10x+11y-12z=13. \end{cases}$

PROBLEMS LEADING TO SYSTEMS OF EQUATIONS INVOLVING THREE OR MORE UNKNOWN QUANTITIES

1. A and B can do a piece of work in 1 hour and 10 minutes, A and C in 1 hour and 24 minutes, and B and C in 2 hours and 20 minutes. How long would it take each to do the work alone?

2. Three pumps are forcing water into a reservoir. The first and second can fill it in a hours, the first and third in b hours, and the second and third in c hours. How long would it take each alone to fill it?

3. A number is represented by three digits. The digit in hundreds' place minus the one in tens' place equals $\frac{2}{3}$ of the one in units' place; the number itself divided by 4 times the hundreds' digit plus 3 times the tens' plus twice the units' equals $15\frac{3}{28}$; and if 198 be subtracted from the number, the order of the digits will be reversed. What is the number?

4. Three men, A, B, and C, were asked to contribute to an amusement home for young men. They agreed to give from their cash in hand as follows: A, a sum equal to $\frac{1}{3}$ of B's plus $\frac{1}{10}$ of C's money; B, a sum equal to $\frac{1}{2}$ of A's plus $\frac{1}{5}$ of C's; and C, a sum equal to $\frac{1}{4}$ of A's and B's money together. Afterward, in looking over their money, A found that he had left $\frac{1}{2}$ as much as B, and $\frac{2}{15}$ as much as C, and B found that he had $\frac{4}{15}$ as much as C. How much had each in hand at first if C had left $375?

5. A and B can do a piece of work in m days; B and C can do it in n days; C and D in p days; D in r days. In how many days can all working together do it?

6. A and B start together to ascend a mountain. A should reach the summit half an hour before B; but missing his way he goes a mile and back again, during which he walks at twice his former pace, and then reaches the top 6 minutes before B. C starts 20 minutes after B, and, walking at the rate of $\frac{15}{7}$ miles per hour, reaches the summit 10 minutes after B. If x is the distance up the mountain and y is A's rate, find B's rate.

7. Robert offers to run 3 times around a track while Stuart runs twice around, but he has finished only 150 yards of his third round when Stuart wins. He then offers to run 4 times around for Stuart's 3 times, and quickens his pace in the ratio of 4 : 3. Stuart also quickens his in the ratio of 9 : 8, but in the second round falls off to his pace in the first race, and in the third round goes only 9 yards for 10 he went in the first race. Robert wins by 180 yards. Determine the length of the track.

8. Five men, A, B, C, D, E, agreed that A should give to each of the other four as much money as each of the four already had; then that B should give to the other four as much as each of them then had, and so on. When E had given his portion, it was found that the money was equally divided. If each man then had $32, how much money had he in his possession before the division was made?

CHAPTER XI

INVOLUTION

346. A **Power** is a product arising from using a number one or more times as a factor.

347. To **affect** a quantity with an exponent is to perform upon it the operations indicated by that exponent.

348. **Involution** is the process of affecting a quantity with any exponent.

349. **Laws of Positive Integral Exponents.** Let m and n be any positive integers; then

1. The nth power of the mth power of any quantity is the mnth power of that quantity; that is,

$$(a^m)^n = a^{mn}.$$

Dem. $(a^m)^n = a^m \cdot a^m \cdot a^m \cdot \cdots$ to n factors, by definition.

$$= a^{m+m+m+\cdots \text{ to } n \text{ terms}}, \text{ by Index Law,}$$

$$= a^{mn}. \quad \text{Why?}$$

$\therefore \quad (a^m)^n = a^{mn}.$

2. The nth power of the product of any number of factors equals the product of the nth powers of those factors; that is,

$$(abc \cdots)^n = a^n b^n c^n \cdots.$$

Dem. $(abc\cdots)^n=(abc\cdots)(abc\cdots)\cdots$ to n factors. Why?

$=(a\cdot a\cdot\cdots$ to n factors$)\cdot(b\cdot b\cdot\cdots$ to n factors$)\cdot\cdots$, by Law of Order,

$=a^n\cdot b^n\cdot c^n\cdot\cdots$, by definition.

$\therefore\ (abc\cdots)^n=a^nb^nc^n\cdots\cdots.$

3. The nth power of the quotient of any two quantities equals the quotient of the nth powers of those quantities; that is,

$$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}.$$

Dem. $\left(\frac{a}{b}\right)^n=\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdot\cdots$ to n factors. Why?

$=\dfrac{a\cdot a\cdot a\cdot\cdots \text{ to } n \text{ factors}}{b\cdot b\cdot b\cdot\cdots \text{ to } n \text{ factors}}.$ Why?

$=\dfrac{a^n}{b^n}.$ Why?

$\therefore\ \left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}.$

350. Law of Signs. Any power of a positive quantity is positive. Any even power of a negative quantity is positive, and any odd power is negative; that is,

1. $(+a)^n=+a^n$ whether n is odd or even. Why?
2. $(-a)^n=+a^n$ when n is even. Why?
3. $(-a)^n=-a^n$ when n is odd. Why?

351. PROBLEM 1. **To raise an expression to the nth power by multiplication.**

Rule. *Multiply the expression by itself $n-1$ times.*

Dem. Let the student give the demonstration.

EXAMPLES

Perform by actual multiplication:

1. $(-5x^3)^3$.	7. $(\frac{2}{3}x^2y^3z)^5$.	13. $(a+b)^3$.
2. $(10ab^2c^3)^4$.	8. $(\frac{2}{3}-\frac{3}{4})^2$.	14. $(a-b)^3$.
3. $(2+3+4)^2$.	9. $(a+b-c-d)^2$.	15. $(2x-3y)^3$.
4. $(2+4-3)^3$.	10. $(a-b+c-d)^3$.	16. $(2x-3y+4z)^2$.
5. $(a-b+c)^2$.	11. $(4-3+2-1)^2$.	17. $(3x-4y+z)^3$.
6. $(a+b-c)^3$.	12. $(4+3-2-1)^3$.	18. $(a+b+c+d+e+f)^2$.

352. PROBLEM 2. **To affect a monomial with any positive integral exponent by the Law of Exponents and of Signs for Involution.**

Rule. *Multiply the exponent of each factor by the given exponent, observing the Law of Signs for Involution.*

Dem. Laws 1, 2, 3, and 4 for Involution.

MODEL SOLUTION

Operation. $(-2a^2b^8c^4)^5 = -2^5a^{10}b^{15}c^{20}$.

Proof. $(-2a^2b^8c^4)^5 = (-2a^2b^8c^4)\cdot(-2a^2b^8c^4)\cdot(-2a^2b^8c^4)\cdots$ to 5 factors, by definition,

$= (-2\cdot -2\cdot \cdots$ to 5 factors$)\cdot(a^2\cdot a^2\cdots$ to 5 factors$)\cdot\cdots$, by Law of Order,

$= (-2)^5\cdot(a^2)^5\cdot(b^8)^5\cdot(c^4)^5$, by definition,

$= -2^5a^{10}b^{15}c^{20}$, by Law of Signs and of Exponents.

EXAMPLES

Expand by the laws of Involution:

1. $(abc)^3$.	3. $(-a^2bc)^4$.	5. $(-2x^2y^3z^4)^3$.	7. $(3a^2bc^5)^5$.
2. $(-2mx^2y^3)^4$.	4. $(-10x^5y^{10}z^3)^2$.	6. $(-2ab^2cx^3)^5$.	8. $(-xy^2z^3)^{20}$.

9. $(a^2b^5c^{10})^{200}$. 12. $(ab^2c^3)^n$. 15. $(\frac{1}{2}s^3tu^2)^3$. 18. $(-\frac{2}{3}p^2qr^7)^4$.

10. $(-as^2x^3)^m$. 13. $(\frac{2}{3}r^2st^3)^n$. 16. $\left(\frac{2a^2xy^3}{3abc^2}\right)^5$. 19. $\left(\frac{-4c^3x}{-5st^2}\right)^3$.

11. $\left(-\frac{x^2y^2z}{ab^2c^3}\right)^{300}$. 14. $\left(\frac{5ax^2}{6bc^n}\right)^m$. 17. $\left(\frac{-a^3dg^2}{+cx^2y^s}\right)^n$. 20. $\left(\frac{+a^rb^sc^t}{-xy^2z^3}\right)^q$.

353. Problem 3. **To expand a binomial having any positive integral exponent.**

Let $a + b$ represent any binomial; then

1. $(a+b)^2 = a^2 + 2ab + b^2$, by Theorem 1.
2. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, by Theorem 7.
3. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, by multiplying (2) by $a + b$.
4. $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, by multiplying (3) by $a + b$.
5. $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$, by multiplying (4) by $a + b$.

By inspecting the expansion of these powers of $a + b$, there may be seen the following:

354. Laws of Coefficients and Exponents.

1. The coefficient of the first term of the expansion is 1; of the second, the exponent of the binomial.

2. The coefficient of any term after the first may be found by multiplying the coefficient of the preceding term by the exponent of the letter of arrangement in that term, and dividing the product by the number of terms thus far obtained.

3. The exponent of the first letter in the expansion is the same as the exponent of the given binomial, and decreases by 1 to the right until it becomes 0.

4. The exponent of the second letter begins with 1 in the second term of the expansion and increases by 1 to the right until it becomes the same as the given exponent.

355. Laws of Verification.

1. The first term contains only the first letter (or term) of the binomial, the last term only the second, and all other terms contain both.

2. There is one more term in the expansion than there are units in the given exponent.

3. The sum of the exponents in each term is equal to the given exponent.

4. The coefficients equally distant from their respective extremes are equal, and hence the coefficients of the last half of the expansion can be written out from the first half in a reverse order.

5. If the sign of the first term of the binomial is positive and the second negative, the signs of the terms in the expansion will alternate, the first being positive.

Model Solution

$$(2m^2-3n^3)^4=(2m^2)^4+4(2m^2)^3(-3n^3)+6(2m^2)^2(-3n^3)^2+4(2m^2)(-3n^3)^3+(-3n^3)^4$$

$$=16m^8-96m^6n^3+216m^4n^6-216m^2n^9+81n^{12}.$$

356. The General Formula or Binomial Theorem.

If $(a+b)^5$ be expanded by the foregoing laws and the operations be indicated instead of worked out, it will be seen that

$$(a+b)^5 = a^5 + \frac{5}{1}a^{5-1}b + \frac{5(5-1)}{1\cdot 2}a^{5-2}b^2 + \frac{5(5-1)(5-2)}{1\cdot 2\cdot 3}a^{5-3}b^3$$
$$+ \frac{5(5-1)(5-2)(5-3)}{1\cdot 2\cdot 3\cdot 4}a^{5-4}b^4$$
$$+ \frac{5(5-1)(5-2)(5-3)(5-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^{5-5}b^5.$$

Put $5 = n$ and let r be a positive integer not greater than n.

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3$$
$$+ \cdots + \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdot 4\cdot \cdots \cdot r}a^{n-r}b^r$$
$$+ \cdots + \frac{n(n-1)}{1\cdot 2}a^2b^{n-2} + \frac{n}{1}ab^{n-1} + b^n, \qquad \textbf{the Formula.}$$

The term $+\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdot \cdots \cdot r}a^{n-r}b^r$ is called the *general term*, and can be made to assume in succession the form and value of every term in the series after the first by giving to r the successive values $1, 2, 3, 4, 5, \cdots n-2, n-1, n$.

It will be noticed that in the second term $r = 1$, in the third 2, etc.; hence $r =$ one less than the number of any particular term under consideration.

Model Solution

Find the 4th term of $(2x - 3y^2)^6$.

$n = 6$	$\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdot \cdots \cdot r}a^{n-r}b^r.$
$r = 3$	$= \dfrac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}(2x)^3(-3y^2)^3$
$n - r + 1 = 4$	$= 5\cdot 4\cdot 2^3x^3\cdot -3^3y^6$
$a = 2x$	$= -2^5\cdot 3^3\cdot 5x^3y^6$
$b = -3y^2$	$= -4320x^3y^6.$

EXAMPLES

1. $(x-y)^5$.
2. $(x+y)^6$.
3. $(2x^2-3y^3)^3$.
4. $(x-y)^7$.
5. $(m-n)^{10}$.
6. $(x+1)^5$.
7. $(2-x)^5$.
8. $(3-y^2)^4$.
9. $(1-y)^1$.
10. $(1-x^2)^0$.
11. $(1-z)^5$.
12. $(1-xy^2)^4$.
13. $(2a^2-5b^4)^4$.
14. $(3ab^2-4c^3d^4)^3$.
15. $(m^2+a^3)^5$.
16. $(7xy^3-st^2)^3$.
17. $(6-4)^3$.
18. $(3-0)^4$.
19. $(1+1)^5$.
20. $(\frac{1}{2}x-\frac{1}{3}y)^4$.
21. $(\frac{2}{3}a^3b-\frac{3}{4}xy^2)^3$.
22. $\left(\dfrac{a}{b}+\dfrac{c}{d}\right)^4$.
23. $\left(\dfrac{a}{x}-\dfrac{2}{y}\right)^5$.
24. $\left(\dfrac{2a}{3x}-\dfrac{3x}{2a}\right)^4$.
25. $(1+\frac{2}{3})^5$.
26. $\left(\dfrac{a}{2x}-1\right)^4$.
27. $(x-y)^{100}$.
28. $(x+y)^n$.
29. $(y+1)^m$.
30. $(1-x)^r$.
31. Check Ex. 16.
32. Check Ex. 24.
33. Find the 4th term of $(x+y)^7$.
34. Find the 6th term of $(a-b)^8$.
35. Find the 3d term of $(x-y)^{100}$.
36. Find the 5th term of $(a+b)^4$.
37. Find the 4th term of $(2x^3-3y^5)^5$.
38. Find the 5th term of $(\frac{1}{2}x^2-\frac{2}{3}y^3)^5$.
39. Find the 2d term of $(2x-3y)^7$.
40. Find the middle term of $(x-2y)^6$.
41. Find the two middle terms of $(\frac{1}{2}x-2z)^5$.
42. Find the two middle terms of $(3a-\frac{1}{3}b)^7$.

357. Cor. Any polynomial may be written as a binomial and expanded by the Binomial Theorem.

Model Solution

$(a+b+c+d)^4 = \{(a+b)+(c+d)\}^4$

$= (a+b)^4 + 4(a+b)^3(c+d) + 6(a+b)^2(c+d)^2 + 4(a+b)(c+d)^3 + (c+d)^4$

$= a^4 + b^4 + c^4 + d^4 + 4\,a^3b + 4\,ab^3 + 4\,a^3c + 4\,ac^3 + 4\,a^3d + 4\,ad^3$

$+ 4\,b^3c + 4\,bc^3 + 4\,b^3d + 4\,bd^3 + 4\,c^3d + 4\,cd^3 + 6\,a^2b^2 + 6\,a^2c^2$

$+ 6\,a^2d^2 + 6\,b^2c^2 + 6\,b^2d^2 + 6\,c^2d^2 + 12\,a^2bc + 12\,ab^2c$

$+ 12\,abc^2 + 12\,a^2bd + 12\,ab^2d + 12\,abd^2 + 12\,a^2cd + 12\,ac^2d$

$+ 12\,acd^2 + 12\,b^2cd + 12\,bc^2d + 12\,bcd^2 + 24\,abcd.$

Note. This result can be, and in fact was, written out immediately by (5) in the following

Special Forms

1. $(a \pm b)^2 = a^2 + b^2 \pm 2\,ab$, by Special Theorems 1 and 2.
2. $(a \pm b)^3 = a^3 \pm b^3 \pm 3\,a^2b + 3\,ab^2$, by Special Theorem 7.
3. $(a + b + c + \cdots)^2 = \Sigma a^2 + \Sigma 2\,ab$, by Special Theorem 6.
4. $(a + b + c + \cdots)^3 = \Sigma a^3 + \Sigma 3\,a^2b + \Sigma 6\,abc$, by Special Theorem 8.
5. $(a + b + c + \cdots)^4 = \Sigma a^4 + \Sigma 4\,a^3b + \Sigma 6\,a^2b^2 + \Sigma 12\,a^2bc + \Sigma 24\,abcd.$
6. $(a + b + c + \cdots)^4 = \{(a + b + c + \cdots)^2\}^2.$
7. $(a + b + c + \cdots)^6 = \{(a + b + c + \cdots)^3\}^2.$

EXAMPLES

1. $(a+b+c)^2$.
2. $(a+b+c)^3$.
3. $(a+b+c)^4$.
4. $(a-b+c-d)^3$.
5. $(2\,x-3\,y^2+4\,z^3)^4$.
6. $(a-b+c-d)^4$.
7. $(a+b+c)^6$.
8. $(a+b+c+d+e+f+g)^2$.
9. $(a+b+c+d+e)^3$.
10. $(3\,x^2-y+2\,z^3)^3$.
11. $(2\,x-3\,y^2+s^3-2\,t^4)^2$.
12. $(\frac{1}{2}\,x-\frac{1}{3}\,y+\frac{3}{4}\,z)^2$.
13. $\left(\dfrac{a}{b}+\dfrac{c}{d}-\dfrac{e}{f}+\dfrac{g}{h}\right)^2$.
14. $\left(\dfrac{2\,a}{3\,b^2}-\dfrac{c^3}{2\,d^2}-\dfrac{e}{f}\right)^3$.
15. Check Exs. 13 and 14.

CHAPTER XII

EVOLUTION

358. What is a root of a number? The degree of a root? What is the degree of $\sqrt[3]{8}$?

359. Evolution is the process of extracting roots of numbers. What is the radical sign? How is any root expressed?

360. A **Radical Number** is an indicated root of a number. Give an illustration.

361. An **Even Root** is one whose index is an even number. Give an illustration.

362. An **Odd Root** is one whose index is an odd number. Give an illustration.

363. An **Imaginary Quantity** is an indicated even root of a negative quantity; as, $\sqrt{-4}$, $\sqrt[4]{-3}$.

Does $\sqrt{-4} = -2$? Why not? Does $\sqrt[3]{-8} = -2$? Explain. Does $\sqrt{-4} = 2\sqrt{-1}$?

364. A **Real Quantity** is one that is not imaginary.

365. Laws of Evolution.

1. A root of a power of any quantity is that quantity with an exponent equal to the index of the power divided by the index of the root; that is,

$$\sqrt[n]{a^{mn}} = a^{\frac{mn}{n}} = a^m.$$

Dem. $a^{mn} = (a^m)^n = n$ factors of a^m, by law 1 of Involution.
But the nth root of a quantity is one of its n equal factors.

$$\therefore\ \sqrt[n]{a^{mn}} = a^{\frac{mn}{n}} = a^m.$$

366. Cor. From law 1 it is evident that the numerator of a fractional exponent indicates a power, and the denominator a root.

2. The nth root of the product of any number of factors is equal to the product of the nth roots of those factors; that is,

$$\sqrt[n]{a^nb^nc^n\cdots} = \sqrt[n]{a^n}\cdot\sqrt[n]{b^n}\cdot\sqrt[n]{c^n}\cdots = abc\cdots.$$

Dem. $\sqrt[n]{a^nb^nc^n\cdots} = \sqrt[n]{(abc\cdots)^n}$, by law 2 of Involution.

$= abc\cdots.$ Why?

But $\sqrt[n]{a^n}\cdot\sqrt[n]{b^n}\cdot\sqrt[n]{c^n}\cdots = abc\cdots.$ Why?

$\therefore\ \sqrt[n]{a^nb^nc^n\cdots} = \sqrt[n]{a^n}\cdot\sqrt[n]{b^n}\cdot\sqrt[n]{c^n}\cdots.$ Why?

3. The nth root of the quotient of two quantities is equal to the quotient of their nth roots; that is,

$$\sqrt[n]{\frac{a^n}{b^n}} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}} = \frac{a}{b}.$$

Dem. $\sqrt[n]{\frac{a^n}{b^n}} = \sqrt[n]{\left(\frac{a}{b}\right)^n}$ (Why?) $= \frac{a}{b}.$ Why?

But $\frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}} = \frac{a}{b}.$ Why?

$\therefore\ \sqrt[n]{\frac{a^n}{b^n}} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}}.$ Why?

4. (*a*) Every positive quantity has two real even roots which are numerically equal but with opposite signs; that is,

$\sqrt[n]{a^{mn}} = \pm\, a^m$ when n is even and a^{mn} is positive.

(*b*) Any real quantity has but one real odd root, whose sign is the same as that of the quantity itself; that is,

$$\sqrt[n]{a^{mn}} = + a^m \text{ when } n \text{ is odd and } a^{mn} \text{ is positive,}$$

and $$\sqrt[n]{a^{mn}} = - a^m \text{ when } n \text{ is odd and } a^{mn} \text{ is negative.}$$

Dem. (*a*) is true because an even number of like factors give the same product whether they are positive or negative. (*b*) also is true, — why?

367. Problem 1. **To extract the *n*th root of a perfect *n*th power, by factoring.**

Rule. *Resolve the power into n equal factors, and take one of them for the nth root.*

Dem. Because the *n*th root of a number is one of the *n* equal factors of that number by definition.

Model Solutions

1. Extract the cube root of 27000.

5	27000
5	5400
5	1080
3	216
3	72
3	24
2	8
2	4
	2

$$\begin{aligned}\sqrt[3]{27000} &= \sqrt[3]{(2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 5\cdot 5\cdot 5)}\\ &= \sqrt[3]{\{(2\cdot 3\cdot 5)\cdot(2\cdot 3\cdot 5)\cdot(2\cdot 3\cdot 5)\}}\\ &= 2\cdot 3\cdot 5\\ &= 30.\end{aligned}$$

2. Extract the 4th root of $1296\, a^4b^8x^{16}y^{100}$.

$$\begin{aligned}&\sqrt[4]{1296\, a^4b^8x^{16}y^{100}}\\ &= \sqrt[4]{(2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 3\cdot a\cdot a\cdot a\cdot a\cdot b^2\cdot b^2\cdot b^2\cdot b^2\cdot x^4\cdot x^4\cdot x^4\cdot x^4\cdot y^{25}\cdot y^{25}\cdot y^{25}\cdot y^{25})}\\ &= 2\cdot 3\cdot a\cdot b^2\cdot x^4\cdot y^{25}\\ &= 6\, ab^2x^4y^{25}.\end{aligned}$$

3. Extract the square root of $x^4 + 10x^3 + 11x^2 - 70x + 49$.

$$\sqrt{(x^4 + 10x^3 + 11x^2 - 70x + 49)} = \sqrt{\{(x^2 + 5x - 7)(x^2 + 5x - 7)\}}$$
$$= x^2 + 5x - 7.$$

4. Extract the cube root of $a^6 + 6a^5 - 40a^3 + 96a - 64$.

$\sqrt[3]{a^6} = a^2 =$ 1st term of the root.

Since $6a^5 = 3$ times the square of the first term of the root multiplied by the second term, the second term of the root is $6a^5 \div 3(a^2)^2 = 2a$. $\sqrt[3]{-64} = -4 =$ the last term of the root.

$$\therefore \sqrt[3]{(a^6 + 6a^5 - 40a^3 + 96a - 64)} = \sqrt[3]{(a^2 + 2a - 4)^3} = a^2 + 2a - 4.$$

EXAMPLES

Extract the required root by factoring :

1. $\sqrt{441}$.
2. $\sqrt{81a^4x^2y^6z^8}$.
3. $\sqrt{144a^4m^{6+4a}}$.
4. $\sqrt[3]{1728x^{12a}y^{3b}}$.
5. $\sqrt[4]{9^2m^{8-4a}x^{12}}$.
6. $\sqrt[5]{\dfrac{32m^{2a+3a}}{243m^{3a+2a}}}$.
7. $\sqrt{(a^2 + 2ab + b^2)}$.
8. $\sqrt{2601}$.
9. $\sqrt{25a^4b^6}$.
10. $\sqrt{64a^6x^2y^{2a}}$.
11. $\sqrt[5]{-32m^5y^{20}}$.
12. $\sqrt[3]{26198073}$.
13. $\sqrt[3]{\dfrac{27a^{6-15a}}{64m^3x^{15a}}}$.
14. $\sqrt[3]{a^3b^3(a-b)^3}$.
15. $\sqrt[10]{x^{10a}y^{30b}z^{40m-10}}$.
16. $\sqrt[3]{\dfrac{27a^6b^9}{64x^{12}y^{15}}}$.
17. $\sqrt[3]{125a^{21}y^{51+6n}}$.
18. $\sqrt{(25a^4b^{10}x^{4+4S})}$.
19. $\sqrt[3]{\{-343x^{17+7}y^{6Q}\}}$.
20. $\sqrt{x^2 + 12x + 36}$.
21. $\sqrt[5]{[243a^{10-5m-10n}b^5]}$.
22. $\sqrt[3]{a^6(a^3 + 3a^2b + 3ab^2 + b^3)}$.
23. $\sqrt[3]{-8^{2a}(x^3)^2(y^3 - 3y^2 + 3y - 1)}$.
24. $\sqrt{(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)}$.
25. $\sqrt{x^2 - 32x + 256}$.
26. $\sqrt{4a^2 + 9x^2 + 12ax}$.
27. $\sqrt{36x^6 + 24x^3y + 4y^2}$.
28. $\sqrt{y^2z^2(16x^2 + 9z^2 + 24xz)}$.
29. $\sqrt{(x^4 - 6x^3 + 19x^2 - 30x + 25)}$.
30. $\sqrt{4x^4 - 4x^3 + 5x^2 - 2x + 1}$.

31. $\sqrt{\left(x^2+2x-1-\frac{2}{x}+\frac{1}{x^2}\right)}$.

32. $\sqrt[3]{\left\{x^3+3m^2-5+\frac{3}{m^2}-\frac{1}{m^3}\right\}}$.

33. $\sqrt{x^4-10x^3+39x^2-70x+49}$.

34. $\sqrt[3]{\{-27x^6+81x^5-54x^4-27x^3+18x^2+9x+1\}}$.

368. PROBLEM 2. **To extract the *n*th root of monomials by the laws of Evolution.**

Rule. *Extract the nth root of the numerical coefficient by Problem* 1, *and divide the exponent of each letter by* n.

Dem. See the laws of Evolution.

MODEL SOLUTION

$$\sqrt[3]{27\,a^3b^{3m-6n}c^{12a}} = 3^{\frac{3}{3}}a^{\frac{3}{3}}b^{\frac{3m-6n}{3}}c^{\frac{12a}{3}} = 3\,ab^{m-2n}c^{4a}.$$

EXAMPLES

Extract the required root by the laws of Evolution:

1. $\sqrt{36a^2b^4}$.
2. $\sqrt[4]{16\,a^4b^{16}c^8}$.
3. $\sqrt{\frac{1}{16}\,a^{4m}b^{10n-4s}}$.
4. $\sqrt[s]{x^{2s}y^{3s^2}}$. Check.
5. $\sqrt[6]{\frac{729}{262144}\,a^{12}b^{24}c^{30}}$.
6. $\sqrt[mn]{x^{m^2n}y^{2mn}z^{3mn^2}}$.
7. $\sqrt[100c]{a^{100c}b^{100c^2}d^{100cq}}$.
8. $\sqrt{9\,a^2b^4c^6}$.
9. $\sqrt[3]{-27\,x^3y^9z^6}$.
10. $\sqrt[3]{-8\,x^6y^{39}}$.
11. $\sqrt[3]{\frac{64\,x^3y^6}{27\,w^{12}z^{21}}}$.
12. $\sqrt[4]{\frac{16\,a^4b^{12}}{625\,y^8}}$.
13. $\sqrt[2m]{x^{2m}y^{6mn}}$.
14. $\sqrt[Q]{x^{3Qs-2Q}}$.
15. $\sqrt[5]{-32\,a^5b^{10}}$.
16. $\sqrt[3]{-27\,a^{6m}x^{9m}}$.
17. $\sqrt[5]{\frac{32}{243}\,x^{10}y^{20a}z^{10\;5b}}$.
18. $\sqrt[n]{-2^na^{10n}b^{15n}}$.
19. $\sqrt[st]{x^{2st}y^{st^2}}$. Check.
20. $\sqrt[rt]{a^{2rt}y^{nr^2t}z^{Pqr^2t}}$.
21. $\sqrt[a]{x^ay^{a^2bc}z^{a^3b^3c^3}}$.

369. PROBLEM 3. **To extract the square root of any number.**

Since $(a+b)^2 \equiv a^2 + 2ab + b^2 \equiv a^2 + (2a+b)b$ represents the perfect square of any number, it may be used as a model for extracting the square root of any number by making use of the following assumption: that a shall represent that part of the root already found, and b the next figure in the root sought.

MODEL SOLUTIONS

1. Let it be required to extract the square root of $a^2 + 2ab + b^2$, knowing that its square root is $a+b$.

$$\begin{array}{l|l}
 & a^2 + 2ab + b^2 \;\underline{|\, a + b} \\
 & a^2 \\
\hline
2a & 2ab + b^2 \\
\quad + b & \\
\hline
2a + b & 2ab + b^2 \\
\hline
\end{array}$$

Explanation. Since the left-hand term contains the square of the first figure in the root, such figure can be obtained by extracting the square root of the left-hand term. This gives a for the first figure in the root. What is left after subtracting the square of a from $a^2+2ab+b^2$ is $2ab + b^2$. This is seen to be composed of two factors, $2a + b$ and b. If one of these factors is given, the other can be obtained by dividing $2ab+b^2$ by it. Neither factor is in the root already found. However, a, which is part of $2a + b$, is there and can be made $2a$ by doubling it. With this partial factor $2a$, it may be possible to find the other factor by dividing $2ab + b^2$ by it. Dividing $2ab + b^2$ by $2a$ gives b, which is recognized as the next figure in the root.

Since b is now obtained, the partial factor $2a$ may be made the complete factor $2a + b$ by adding b to $2a$. And since $2a + b$ is one factor and b the other, the number $2ab + b^2$ is obtained by multiplying together these two factors. Subtracting gives a remainder of 0, which shows that $a^2 + 2ab + b^2$ is a perfect square, the square root of which is $a + b$.

2. Extract the square root of 56169, using the method explained in model solution No. 1.

Working Model. $(a+b)^2 \equiv a^2 + 2\,ab + b^2 \equiv a^2 + (2\,a+b)b$, in which a stands for that part of the root already found, and b for the next figure in the root sought.

$$
\begin{array}{lcr|l}
 & & & \quad \underline{a} \\
 & & & \quad a\,b\,b \\
 & & 5'61'69 & \underline{\,2\,3\,7\,} \\
a^2 = 2^2 & = & 4 & \\
\hline
2\,a = 2(20) & = & 40 & 161 \text{ cor. to } 2\,ab + b^2 \\
\underline{b} & = & \underline{3} & \\
2\,a + b & = & 43 & \underline{129 = (2\,a+b)b} \\
\hline
2\,a = 2(230) & = & 460 & 3269 \text{ cor. to } 2\,ab + b^2 \\
\underline{b} & = & \underline{7} & \\
2\,a + b & = & 467 & \underline{3269 = (2\,a+b)b} \\
\hline
\end{array}
$$

Explanation. 1. Point off the number into periods of two figures each, beginning at units, since the square of a number contains twice as many figures as its square root, or twice as many less one; and the number of periods thus pointed off indicates the number of figures in the root.

Let the student show these statements to be true.

2. Since the left-hand period always contains the square of the first figure in the root, such figure is obtained by extracting the square root of the greatest square in the left-hand period. In this case the largest square is 4, the square root of which is 2. The 2 corresponds to a, and 4 to a^2. Subtracting the 4 from the left-hand period gives a remainder of 1, which shows that 5 or (50000) is not a perfect square.

3. This remainder, 1, of the first period equals 100 of the second, which becomes 161 when 61 is added. One period at a time is brought down (in this case 61) because the square of any digit does not exceed two places, and hence one period of two figures is sufficient to yield one figure in the root.

4. Since 161 corresponds to $2\,ab + b^2$, and since $2\,ab + b^2$ is composed of two factors, $2\,a + b$ and b, 161 is assumed to be composed of two factors (if 56100 is a perfect square), one of which will correspond to $2\,a + b$ and the other to b. Now, if we had one of these factors, we could obtain the other by dividing 161 by it. Neither factor is in the root already found. However, $2(=a)$, which is a part of $2\,a + b$, is there and may be made $2\,a$ by doubling it, giving 4. With this partial factor, or

trial divisor, we must try to obtain the next figure in the root and complete the divisor.

5. But since we are dealing with the next lower period which yields the next lower order in the root, this trial divisor must be reduced to the next lower order. This is done by multiplying it by 10, giving 40 for the trial divisor.

6. It is called *trial* divisor because it is only part of a factor, or divisor, and consequently may produce too large a quotient. Hence it is only by trial that the true quotient, or next root figure, is found.

7. The next figure in the root can now be found by dividing 161 by $40 = 3 = b$; for if one factor (partial factor in this case) and the number are given, the other factor can be found by dividing the number by the given factor.

8. Since the next figure in the root, or b, has been found, and since the complete first factor, or divisor, should be $2a + b$, the trial divisor may be made into a complete divisor by adding b, or 3, to it.

9. Since the two factors $(2a + b) = 43$ and $b = 3$ have been found, their product should be the given number 161 (if 561 is a perfect square).

10. Subtracting 129 from 161 gives a remainder of 32, which shows that 56100 is not a perfect square.

Proceeding exactly as before and according to the same reasoning, 32 is reduced to the next lower period and 69 added. a now $= 23$, which, reduced to the next lower, equals 230, and multiplied by $2 = 460 = 2a$. Dividing 3269 by $460 = 7 = b$. Completing the divisor gives $467 = 2a + b$.

Multiplying the two factors together gives 3269, and subtracting leaves no remainder. Hence 56169 is a perfect square, the square root of which is 237.

Note. Instead of letting a represent different numbers, and also b, a' and b', a'' and c'', etc., may be used to represent the different numbers. Work an example using a, a', a'', b, b', b''. In the model solution why does 161 correspond to $2ab + b^2$ but does not equal it?

Questions for Any Root Scheme

1. How are the numbers pointed off into periods, and why?
2. How is the first figure in the root found, and why?
3. How many periods are brought down at a time, and why?
4. How is the trial divisor found, and why?
5. How is the trial divisor reduced to the next lower order, and why reduced?

6. Why is the trial divisor so called?
7. How is the second figure in the root found, and why?
8. How is the divisor completed, and why?
9. What is done with the complete divisor, and why?
10. No remainder shows what, and why?

Note. The answers to the above set of questions constitute a Rule and a Demonstration for any root scheme.

The student should learn to answer them separately and collectively. The answers to the question *how* constitute a rule; and those to *why*, a demonstration.

It is believed that this method is an improvement upon any requiring the learning of long rules which are never long remembered by the student. If only the student learns to see the rule in the model, he will not be liable soon to forget it.

370. Cor. The square root of a common fraction (reduced to its lowest terms) equals the square root of the numerator divided by the square root of the denominator, by law 3, or the square root of an equivalent decimal fraction.

Model Solutions

1. $\sqrt{\frac{8}{18}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$.
2. $\sqrt{\frac{8}{12}} = \sqrt{\frac{2}{3}} = \sqrt{.666666+} = .816+$.

Where does the pointing off of integral numbers begin? Why?
Where does the pointing off of decimal fractions begin? Why?

EXAMPLES IN DECIMAL NUMBERS

Extract the square root and verify:

1. 841.	**5.** 94249.	**9.** 1040400.	**13.** 49.2804.
2. 3364.	**6.** 606841.	**10.** 7284601.	**14.** 1624251204.
3. 10201.	**7.** 826281.	**11.** .763876.	**15.** 46.80231.
4. 42849.	**8.** 1034289.	**12.** 1.324.	**16.** $4\frac{100}{144}$.

17. $\frac{1008}{1183}$.	**20.** 2.	**23.** 6.	**26.** 10.	**29.** $\frac{2}{3}$.	**32.** $\frac{5}{6}$.
18. 1.	**21.** 3.	**24.** 7.	**27.** 11.	**30.** $\frac{3}{4}$.	**33.** $\frac{7}{9}$.
19. 0.	**22.** 5.	**25.** 8.	**28.** 12.	**31.** $\frac{4}{5}$.	**34.** $\frac{7}{8}$.

Model Solution

Square Root of Polynomials

Extract square root of $6zx^3 - yzx^2 + z^2x^2 + \frac{1}{4}y^2x^2 - 3yx^2 + 9x^4$.

$$
\begin{array}{ll|l}
 & & \underline{|\,3x^2 - \frac{1}{2}xy + xz} \\
 & & 9x^4 - 3x^2y + \frac{1}{4}x^2y^2 + 6x^3z - x^2yz + x^2z^2 \\
 & a^2 = (3x^2)^2 = 9x^4 & \\
\hline
2a = 2(3x^2) & = 6x^2 & -3x^3y + \frac{1}{4}x^2y^2 \text{ cor. to } 2ab + b^2 \\
b & = -\frac{1}{2}xy & \\
2a+b & = 6x^2 - \frac{1}{2}xy & -3x^3y + \frac{1}{4}x^2y^2 = (2a+b)b \\
\hline
2a = 2(3x^2 - \frac{1}{2}xy) & = 6x^2 - xy & +6x^3z - x^2yz + x^2z^2 \text{ cor. to } 2ab + b^2 \\
b & = xz & \\
2a+b & = 6x^2 - xy + xz & +6x^3z - x^2yz + x^2z^2 = (2a+b)b \\
\hline
\end{array}
$$

Why are questions 1, 3, and 5 of the set of ten unnecessary in the case of algebraic expressions? Why are decimal numbers put first?

Let the student explain the above example, answering questions 2, 4, 6, 7, 8, 9, 10, of the set of ten on page 271.

EXAMPLES

Extract the square root and verify:

1. $4x^4 - 4x^3 + 1 - 2x + 5x^2$.
2. $12y^3 - 6y + 1 + 4y^4 + 5y^2$.
3. $2a^3bc^2 + a^4c^2 + a^2b^2c^2$.
4. $-2m^4x + m^4 + m^4x^2$.
5. $8x^2 + 4x^3 + 8x + x^4 + 4$.
6. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
7. $x^4 + 39x^2 + 49 - 10x^3 - 70x$.
8. $4x^4 - 12x^3 - 7x^2 + 24x + 16$.
9. $x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz$.
10. $20x^3 + 15x^4 + 15x^2 + 6x^5 + 6x + 1 + x^6$.
11. $16a^4 - 24a^3x + 49a^2x^2 - 30ax^3 + 25x^4$.
12. $x^6 - 6x^5 - 34x^3 + 17x^4 + 46x^2 + 25 - 40x$.
13. $4a^6 - 12a^5b - 11b^2a^4 + 58a^3b^3 + 49b^6 - 70ab^5 - 17a^2b^4$.

14. $x^4 + x^3 + \frac{5}{4}x^2 + \frac{1}{2}x + \frac{1}{4}$.

15. $x^4 + 8x^3 + 18x^2 + 8x + 1$.

16. $9x^2 - 30x + \dfrac{1}{x^2} - \dfrac{10}{x} + 31$.

17. $x^4 + 8x^2 + 24 + \dfrac{16}{x^4} + \dfrac{32}{x^2}$.

18. $a^4b^2 + a^3b + \frac{1}{4}a^2 + a^2b^2 + \frac{1}{2}ab + \frac{1}{4}b^2$.

19. $81x^2 + z^2 + 9y^2 - 54xy - 6yz + 18xz$.

20. $\frac{9}{16}x^2 + \frac{9}{4}y^2 + \frac{49}{16}z^2 - \frac{9}{4}xy + \frac{21}{8}xz - \frac{21}{4}yz$.

21. $25x^2 + 30xy^2 + 9y^4 - 2.5xz - 1.5y^2z + .0625z^2$.

371. PROBLEM 4. **To extract the cube root of any number.**

Since $(a+b)^3 \equiv a^3 + (3a^2 + 3ab + b^2)b$ represents the perfect cube of any number, it may be used as a model for extracting the cube root of any number by making use of the following assumption: that a shall represent that part of the root already found, and b the next figure in the root sought.

MODEL SOLUTIONS

1. Let it be required to extract the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$, knowing the root to be $a + b$.

$$
\begin{array}{l|l|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 & \underline{a+b} \\
 & a^3 & \\
\hline
3a^2 & +3a^2b + 3ab^2 + b^3 & \\
\quad + 3ab & & \\
\qquad + b^2 & & \\
\hline
3a^2 + 3ab + b^2 & +3a^2b + 3ab^2 + b^3 & \\
\hline
\end{array}
$$

Explanation. Since the left-hand term contains the cube of the first figure in the root, the first root figure can be found by extracting the cube root of the left-hand term. This gives a for the first figure in the root. What is left after using a^3 is $3a^2b + 3ab^2 + b^3$. This is seen

to be composed of two factors, one of which is $3a^2 + 3ab + b^2$, and the other b. If one of these factors is given, the other is obtained by dividing $3a^2b + 3ab^2 + b^3$ by it. Neither factor is in the root already found. However, a, which is part of $3a^2 + 3ab + b^2$, is there, and can be made $3a^2$ by squaring it and multiplying the result by 3. $3ab + b^2$ cannot be obtained until the factor b is found. Dividing $3a^2b + 3ab^2 + b^3$ by the partial factor $3a^2$ gives b, which is recognized as the second figure in the root and also the missing factor. But now that b is found, the first factor, or divisor, can be completed by adding $3ab + b^2$ to $3a^2$, which gives $3a^2 + 3ab + b^2$. Since this is one factor and b is the other, their product is taken $= 3a^2b + 3ab^2 + b^3$. Subtracting leaves no remainder. Hence $a^3 + 3a^2b + 3ab^2 + b^3$ is a perfect cube, the cube root of which is $a + b$.

2. Extract the cube root of 43095878216.

Working Model. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + (3a^2 + 3ab + b^2)b$, in which a stands for that part of the root already found, and b for the next figure in the root sought.

$$\begin{array}{l} \underline{a} \\ \underline{a} \\ a\ b\ b\ b \end{array}$$

43'095'878'216 | 3 5 0 6

$a^3 = 3^3 =$ 27

		16095 cor. to $3a^2b + 3ab^2 + b^3$
$3a^2 = 3.(30)^2 =$	2700	
$3ab = 3.(30).5 =$	450	
$b^2 = (5)^2 =$	25	
$3a^2 + 3ab + b^2 =$	3175	$15875 = (3a^2 + 3ab + b^2)b$
$3a^2 = 3(350)^2 =$	367500	220878 cor. to $3a^2b + 3ab^2 + b^3$
$3ab = 3.(350).0 =$	0000	
$b^2 = (0)^2 =$	0	
$3a^2 + 3ab + b^2 =$	367500	$000000 = (3a^2 + 3ab + b^2)b$
$3a^2 = 3(3500) =$	36750000	220878216 cor. to $3a^2b + 3ab^2 + b^3$
$3ab = 3.(3500).6 =$	63000	
$b^2 = (6)^2 =$	36	
$3a^2 + 3ab + b^2 =$	36813036	$220878216 = (3a^2 + 3ab + b^2)b$

Explanation. 1. Separate the number into periods of three figures each, because the cube of any number contains three times as many figures, or three times as many less one or two, as its cube root, and the number of periods will indicate the number of figures in the root.

2. Extract the cube root of the largest cube in the left-hand period for the first figure in the root, because the left-hand period always contains the cube of the first figure in the root. In this case the largest cube in 43 is 27, the cube root of which is 3. 3 corresponds to a, and 27 to a^3. Subtracting this cube from 43 leaves the remainder 16.

3. Bring down one new period at a time, because the cube of any digit never exceeds one period of three figures, and hence one period is sufficient to yield the next figure in the root.

4. Since 16095 corresponds to $3a^2b+3ab^2+b^3$, and since the latter is made up of two factors, 16095 is assumed to be made up of two factors (and is, if 43095 is a perfect cube), one of which will correspond to $3a^2+3ab+b^2$, and the other to b. Now, if we had one of these factors, we could get the other by dividing the number by that factor. There is neither factor in the root already found. However, 3 ($=a$), which is a part of $3a^2+3ab+b^2$, is there and can be made $3a^2$ by squaring 3 and then multiplying by 3. With this partial factor (or divisor) we must try to obtain the other factor, which will be the next figure in the root, and complete the first factor (or divisor).

5. But since we are dealing with the next lower period which yields the next lower order in the root, a must first be reduced to the next lower order. This is done by multiplying it by 10. This result must now be squared and then multiplied by 3. This gives 2700 for the partial factor (or trial divisor).

6. It is called *trial* divisor because it is only part of a factor, or divisor, and consequently may produce too large a quotient. Hence, it is only by trial that the true quotient, or next root figure, is found.

7. The next figure in the root can now be found by dividing 16095 by $2700 = 5 = b$; for if we have one factor (partial) and the number given, we can get the other by dividing the number by the given factor.

8. Since the next root figure has been found, the complete first factor, or $3a^2+3ab+b^2$, can be obtained by adding $3ab+b^2(=450+25)$ to $3a^2(=2700)$.

9. Since we have both factors, $3a^2+3ab+b^2(=3175)$ and $b(=5)$, we can get the number by multiplying them together $= 15875$.

10. Subtracting 15875 from 16095 gives a remainder of 220, which shows that 43095 is not a perfect cube.

Continuing the process by the same method and according to the same reasoning, it is found that 43095878216 is a perfect cube, the cube root of which is 3506.

372. Cor. The cube root of a common fraction (reduced to its lowest terms) is the cube root of the numerator divided by the cube root of the denominator (law 3), or the cube root of an equivalent decimal fraction.

EXAMPLES IN DECIMAL NUMBERS

Extract the cube root and verify:

1. 6859.
2. 21952.
3. 195112.
4. 1124864.
5. 2000376.
6. 9129329.
7. 13824000.
8. 45882712.
9. 151419437.
10. 1058089859.
11. 1033364331.
12. 1030301000.
13. 900.
14. 879.
15. 415.
16. $\frac{2}{3}$.
17. $\frac{5}{6}$.
18. $\frac{3}{4}$.
19. $\frac{7}{8}$.
20. $\frac{5}{27}$.
21. $\frac{54}{248}$.
22. $\frac{192}{375}$.
23. $2\frac{10}{27}$.
24. $2 - \frac{307}{216}$.

Model Solution

Cube Root of Polynomials

Extract the cube root of $x^6 - 9x^4y^2 + 27x^2y^4 - 27y^6$.

$$
\begin{array}{rcl|l}
 & & & x^6 - 9x^4y^2 + 27x^2y^4 - 27y^6 \;\underline{|\, x^2 - 3y^2} \\
 & & a^3 = (x^2)^3 = x^6 & \\
\hline
3a^2 = 3(x^2)^2 & = & 3x^4 & -9x^4y^2 + 27x^2y^4 - 27y^6 \\
3ab = 3(x^2)(-3y^2) & = & -9x^2y^2 & \\
b^2 = (-3y^2)^2 & = & 9y^4 & \\
\hline
3a^2 + 3ab + b^2 & = & 3x^4 - 9x^2y^2 + y^4 & -9x^4y^2 + 27x^2y^4 - 27y^6 \\
\hline
\end{array}
$$

EXAMPLES

Extract the cube root and verify:

1. $36ab^5 - 8b^6 - 54a^2b^4 + 27a^3b^3$.
2. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.
3. $x^6 + 6x^5 + 21x^4 + 44x^3 + 63x^2 + 54x + 27$.
4. $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

5. $x^6+93x^4+558x^2+216-15x^5-305x^3-540x$.

6. $8x^6+36x^5y+66x^4y^2+63x^3y^3+33x^2y^4+9xy^5+y^6$.

7. $8a^6+48a^5b+60a^4b^2-80a^3b^3-90a^2b^4+108ab^5-27b^6$.

8. $125x^6-225x^5y-15x^4y^2+153x^3y^3+6x^2y^4-36xy^5-8y^6$.

9. $27x^6-54x^5y+171x^4y^2-188x^3y^3+285x^2y^4-150xy^5+125y^6$.

10. $c^6-12c^5b+60b^2c^4-160c^3b^3+240b^4c^2-192cb^5+64b^6$.

11. $8b^6+27a^6-36ab^5-81a^5b+90a^2b^4-135a^3b^3+135a^4b^2$.

12. $a^6-18a^5x+135a^4x^2-540a^3x^3+1215a^2x^4-1458ax^5+729x^6$.

13. $a^3+b^3+c^3+3a^2b+3ab^2+3a^2c+3ac^2+3b^2c+3bc^2+6abc$.

14. $x^3+6x^2y+12xy^2+8y^3+9x^2z+36xyz+36y^2z+27xz^2+54yz^2+27z^3$.

15. $a^3-15a^2x+75ax^2-125x^3+3a^2z-30axz+75x^2z+3az^2-15xz^2+z^3$.

16. $8x^3-36x^2y+54xy^2-27y^3-12x^2z+36xyz-27y^2z+6xz^2-9yz^2-z^3$.

17. $3m^2n^2+1-3mn-m^3n^3$.

18. $1+3x^2-5x^6+3x^{10}-x^{12}$.

19. $\frac{1}{8}x^3-\frac{3}{8}x^2y+\frac{3}{8}xy^2-\frac{1}{8}y^3$.

20. $\frac{1}{64}x^3+\frac{3}{32}x^2y+\frac{3}{16}xy^2+\frac{1}{8}y^3$.

21. $\frac{8}{27}a^3+\frac{4}{3}a^2b+2ab^2+b^3$.

22. $x^3+3x^2-5+\frac{3}{x^2}-\frac{1}{x^3}$.

23. $1-2x$ to 3 terms.

24. $1-x+x^2$ to 3 terms.

25. $x^3-a^3x^2$ to 3 terms.

26. $\frac{8}{a^3}-\frac{12}{a^2}+\frac{18}{a}-13+9a-3a^2+a^3$.

Note. The 4th root is equal to the square root of the square root, and the 6th root is equal to the cube root of the square root.

27. Find the 4th root of $16m^4-96m^3n+216m^2n^2-216mn^3+81n^4$.

28. Find the 6th root of 4826809.

29. Find the 6th root of $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+1$.

30. Find the 4th root of 20736.

CHAPTER XIII

GENERAL THEORY OF EXPONENTS

373. An **Exponent** is any symbol of number written at the right and a little above another symbol of number.

374. The **Five General Laws of Exponents.**

1. The mth power of any quantity multiplied by the nth power of the same quantity equals the $(m+n)$th power of that quantity. Law of Exponents in multiplication.

That is, $$a^m \cdot a^n = a^{m+n}.$$

2. The mth power of any quantity divided by the nth power of that quantity equals the $(m-n)$th power of that quantity. Law E in division.

That is, $$a^m \div a^n = a^{m-n}.$$

3. The nth power of the mth power of any quantity equals the mnth power of that quantity. Law 1 of Evolution.

That is, $$(a^m)^n = a^{mn}.$$

4. The nth power of the product of two or more factors equals the product of the nth powers of those factors. Law 2 of Involution.

That is, $$(a \cdot b \cdot c \cdot \cdots)^n = a^n \cdot b^n \cdot c^n \cdot \cdots.$$

5. The nth power of the quotient of any two quantities equals the quotient of the nth powers of those quantities. Law 3 of Involution.

That is, $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

375. Prop. 1. Any root of any power of a quantity is equal to the same power of the same root of that quantity, provided the arithmetical root is understood.

That is, $$\sqrt[t]{a^s} = (\sqrt[t]{a})^s.$$

Dem. 1. Let $\sqrt[t]{a} = x$.

2. Then $a = x^t$; Why?

3. $a^s = x^{st}$; Why?

4. $\sqrt[t]{a^s} = x^s$. Why?

5. Again $(\sqrt[t]{a})^s = x^s$, from (1). Why?

$\therefore\ \sqrt[t]{a^s} = (\sqrt[t]{a})^s$, for both $= x^s$.

That the arithmetical root must be understood may be seen from the following example: if t is even, $\sqrt[t]{a^t} = \pm a$; but $(\sqrt[t]{a})^t = +a \not\equiv \pm a$.

Hitherto only positive integral exponents have been used. It remains now to become familiar with the use of fractional and negative exponents.

376. The **Fundamental Law for all Exponents** whatsoever is

$$a^m \cdot a^n = a^{m+n}.$$

It is highly essential that algebraic symbols shall always obey the same laws whatever their values may be. If the representative exponents m and n are not positive integers,

the only restriction that must be imposed upon them is that the meaning of a^m and a^n shall in all cases be such that the Fundamental Law of Exponents shall always be true;

That is, $$a^m \cdot a^n = a^{m+n}.$$

This law, therefore, may be made the basis for further operation and discussion. Thus, to find the meaning of $a^{\frac{3}{4}}$ consistent with the Index Law, $a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}}$ must equal $a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^{\frac{3}{4}\cdot 4} = a^3$. Hence $a^{\frac{3}{4}}$ is one of the 4 equal factors of a^3.

377. Problem 1. **To find the meaning of any positive fractional exponent.** See Cor. to law 1 of Evolution.

Let s and t represent any positive integers, and a any number; then

$$a^{\frac{s}{t}} \cdot a^{\frac{s}{t}} \cdot a^{\frac{s}{t}} \cdot \cdots \text{ to } t \text{ factors} = a^{\frac{s}{t}+\frac{s}{t}+\frac{s}{t}+\dots\text{ to } t \text{ terms}}, \text{ by law 1,}$$

$$= a^{\frac{s}{t}\cdot t}. \quad \text{Why?}$$

$$= a^s. \quad \text{Why?}$$

$\therefore a^{\frac{s}{t}}$ equals one of the t equal factors of a^s. But one of the t equal factors of a number is the tth root of that number by definition.

$\therefore a^{\frac{s}{t}} = \sqrt[t]{a^s}$, or $(\sqrt[t]{a})^s$, by Prop. 1.

Hence a **Positive Fractional Exponent** denotes a root of a power, or a power of a root of any quantity; and the numerator indicates the power, and the denominator the root.

378. Problem 2. **To find the meaning of any negative exponent.**

Let s represent any positive quantity, integral or fractional, and a any number.

Then

1. $a^s \cdot a^{-s} = a^{s-s} = a^0 = 1.$ Why?

2. $\therefore\ a^{-s} = \dfrac{1}{a^s}$, by dividing extreme members of (1) by a^s;

3. and $a^s = \dfrac{1}{a^{-s}}$, by dividing extreme members of (1) by a^{-s}.

Hence the **Negative Exponent** of an expression signifies the reciprocal of that expression with the sign of the exponent changed. It therefore denotes a power if it is integral, or a power of a root if fractional, of an expression used as a divisor.

ILLUSTRATION. $5\,a^2x^{-3} = 5\,a^2 \div x^3$, or $5\,a^2/x^3$.

379. COR. 1. The sign of the exponent of any quantity may be changed by changing the sign $\times$ to $\div$, or $\div$ to $\times$, belonging to the quantity.

ILLUSTRATIONS. 1. $a^5 \div a^3 = a^5 \times a^{-3} = a^2.$

2. $a^{\frac{4}{5}} \times a^{-\frac{2}{5}} = a^{\frac{4}{5}} \div a^{\frac{2}{5}} = a^{\frac{2}{5}}.$

The *arithmetical value* of the exponent indicates a power, or a power of a root; and the *sign* of the exponent indicates whether this power is to be used as a multiplier or a divisor. Hence if the sign of the exponent is changed, the quantity becomes a divisor if it was a multiplier before, and *vice versa.*

380. COR. 2. A *factor* may be transferred from either term of a fraction to the other by changing the sign of its exponent.

ILLUSTRATIONS. 1. $\dfrac{a}{b} = ab^{-1}.$ 2. $\dfrac{1}{a-b} = (a-b)^{-1}.$

3. $\dfrac{a+b}{a-b^{-1}} = \dfrac{\dfrac{1}{a^{-1}} + \dfrac{1}{b^{-1}}}{\dfrac{1}{a^{-1}} + \dfrac{1}{b}}.$

4. $\dfrac{a^{-2}b + ab^{-2}}{xy^{-1} - x^{-2}y} = \dfrac{\dfrac{b}{a^2} + \dfrac{a}{b^2}}{\dfrac{x}{y} - \dfrac{y}{x^2}}.$

381. Prop. 2. The nth root of the mth root of any quantity equals the mnth root of that quantity; that is,

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}.$$

Dem. If a quantity is resolved into m equal factors, and each of these is again resolved into n equal factors, the quantity will be resolved into mn equal factors.

382. PROBLEM 3. **To prove the five Laws of Exponents when m and n are positive fractions.**

Let q, r, s, and t represent any positive integers, and let $\frac{q}{r} = m$ and $\frac{s}{t} = n$.

(a) **Dem.** Law 1

Dem. 1

1. $a^{\frac{q}{r}} \cdot a^{\frac{s}{t}} = a^{\frac{qt}{rt}} \cdot a^{\frac{rs}{rt}}$. Why?
2. $\quad = a^{\frac{qt+rs}{rt}}$. Why?

Dem. 2

1. Let $x = a^{\frac{q}{r}} \cdot a^{\frac{s}{t}}$.
2. $x^{rt} = a^{qt} \cdot a^{rs}$. Why?

$\therefore\ x = a^{\frac{qt+rs}{rt}}$. Why?

(b) **Dem.** Law 2

Dem. 1

1. $a^{\frac{q}{r}} \div a^{\frac{s}{t}} = a^{\frac{q}{r}} \cdot a^{-\frac{s}{t}}$. Why?
2. $\quad = a^{\frac{qt-rs}{rt}}$. Why?

Dem. 2

1. Let $x = a^{\frac{q}{r}} \div a^{\frac{s}{t}}$.
2. $x^{rt} = a^{qt} \div a^{rs}$. Why?

$\therefore\ x = a^{\frac{qt-rs}{rt}}$. Why?

(c) **Dem.** Law 3

Dem. 1

1. $(a^{\frac{q}{r}})^{\frac{s}{t}} = \sqrt[t]{(\sqrt[r]{a^q})^s}$. Why?
2. $\quad = \sqrt[t]{\sqrt[r]{a^{qs}}}$. Why?
3. $\quad = \sqrt[rt]{a^{qs}}$. Why?
4. $\quad = a^{\frac{qs}{rt}}$. Why?

Dem. 2

1. Let $x = (a^{\frac{q}{r}})^{\frac{s}{t}}$.
2. $x^t = (a^{\frac{q}{r}})^s$. Why?
3. $x^t = a^{\frac{qs}{r}}$. Why?
4. $x^{rt} = a^{qs}$. Why?

$\therefore\ x = a^{\frac{qs}{rt}}$. Why?

(*d*) **Dem.** Law 4

Dem. 1

1. $(ab)^{\frac{s}{t}} = \sqrt[t]{(ab)^s}$. Why?
2. $= \sqrt[t]{a^s b^s}$. Why?
3. $= \sqrt[t]{a^s} \cdot \sqrt[t]{b^s}$. Why?
4. $= a^{\frac{s}{t}} b^{\frac{s}{t}}$. Why?

Dem. 2

1. Let $x = (ab)^{\frac{s}{t}}$.
2. $x^t = (ab)^s$. Why?
3. $x^t = a^s b^s$. Why?
4. $\therefore\ x = a^{\frac{s}{t}} b^{\frac{s}{t}}$. Why?

(*e*) **Dem.** Law 5

Dem. 1

1. $\left(\frac{a}{b}\right)^{\frac{s}{t}} = \sqrt[t]{\left(\frac{a}{b}\right)^s}$. Why?
2. $= \sqrt[t]{\frac{a^s}{b^s}}$. Why?
3. $= \frac{\sqrt[t]{a^s}}{\sqrt[t]{b^s}}$. Why?
4. $= a^{\frac{s}{t}} / b^{\frac{s}{t}}$. Why?

Dem. 2

1. Let $x = \left(\frac{a}{b}\right)^{\frac{s}{t}}$.
2. $x^t = \left(\frac{a}{b}\right)^s$. Why?
3. $x^t = \frac{a^s}{b^s}$. Why?
4. $\therefore\ x = a^{\frac{s}{t}} / b^{\frac{s}{t}}$. Why?

383. Cor. Multiplying both terms of a fractional exponent by the same number does not change the value of the exponent and hence that of the whole expression.

384. Problem 4. **To prove the five Laws of Exponents when *m* and *n* are negative and either integral or fractional.**

Let s and t be any positive integers or fractions and a any quantity; and let $-s = m$ and $-t = n$.

(*a*) **Dem.** Law 1

Dem. 1

1. $a^{-s} \cdot a^{-t} = \frac{1}{a^s a^t}$. Why?
2. $= \frac{1}{a^{s+t}}$. Why?
3. $= a^{-s-t}$. Why?

Dem. 2

1. Let $x = a^{-s} \cdot a^{-t}$.
2. $a^t x = a^{-s} \cdot a^0$. Why?
3. $x = \frac{a^{-s}}{a^t}$. Why?
4. $\therefore\ x = a^{-s-t}$. Why?

(b) **Dem.** Law 2

Dem. 1

1. $a^{-s} \div a^{-t} = a^{-s} \cdot a^{+t}$. Why?
2. $= a^{-s+t}$. Why?

Dem. 2

1. Let $x = a^{-s} \div a^{-t}$.
2. $a^{-t}x = a^{-s}$. Why?

$\therefore x = a^{-s+t}$. Why?

(c) **Dem.** Law 3

Dem. 1

1. $(a^{-s})^{-t} = \dfrac{1}{(a^{-s})^t}$. Why?
2. $= \dfrac{1}{a^{-st}}$. Why?
3. $= a^{st}$.

Dem. 2

1. Let $x = (a^{-s})^{-t}$.
2. $(a^{-s})^t x = 1$. Why?
3. $a^{-st}x = 1$. Why?

$\therefore x = a^{st}$. Why?

(d) **Dem.** Law 4

Dem. 1

1. $(ab)^{-s} = \dfrac{1}{(ab)^s}$. Why?
2. $= \dfrac{1}{a^s b^s}$. Why?
3. $= a^{-s}b^{-s}$. Why?

Dem. 2

1. Let $x = (ab)^{-s}$.
2. $(ab)^s x = 1$. Why?
3. $x = \dfrac{1}{a^s b^s}$. Why?

$\therefore x = a^{-s}b^{-s}$. Why?

(e) **Dem.** Law 5

Dem. 1

1. $\left(\dfrac{a}{b}\right)^{-s} = \left(\dfrac{b}{a}\right)^s$. Why?
2. $= \dfrac{b^s}{a^s}$. Why?
3. $= \dfrac{a^{-s}}{b^{-s}}$. Why?

Dem. 2

1. Let $x = \left(\dfrac{a}{b}\right)^{-s}$.
2. $\left(\dfrac{a}{b}\right)^s x = 1$. Why?
3. $x = \dfrac{b^s}{a^s}$. Why?

$\therefore x = \dfrac{a^{-s}}{b^{-s}}$. Why?

Application of the Theory of Exponents

NOTATION

MENTAL EXERCISES

(*a*) Free from negative exponents:

1. $5a^{-1}b^2c^{-3}$.
2. $\dfrac{3^{-1}ab^{-2}}{c^{-5}}$.
3. $\dfrac{c-x^{-1}}{a^{-2}+x}$.
4. $2^{-3}xy$.
5. $\dfrac{2a^{-2}}{3b^{-2}}$.
6. $\dfrac{c^2}{5x^{-3}}$.
7. $(5abc)^{-1}$.
8. $\dfrac{1}{3a^{-1}b^{-2}}$.
9. $\dfrac{ax^{-1}z^{-2}}{3y^{-5}}$.
10. $10x^{-5}y^{-2}z^{4}$.
11. $\left(\dfrac{a}{b}\right)^{-\frac{a}{b}}$.
12. $\dfrac{3x^{\frac{1}{3}}y^{-3}}{3^{-1}x^{-\frac{2}{3}}y^{3}}$.
13. $\frac{3}{4}x(\frac{1}{2}x)^{-\frac{2}{3}}$.
14. $\dfrac{3^{-2}}{4^{-1}}(-\frac{3}{2}a)^{-2}$.
15. $\frac{5}{8}(\frac{3}{8}x^{-1}y^2)^{-\frac{2}{3}}$.
16. $-(-\frac{8}{27})^{-\frac{2}{3}}$.
17. $\dfrac{a^{-1}+b^{-1}+c^{-2}}{a^{-1}-b^{-1}+c^{-3}}$.
18. $\dfrac{a^{-2}+a^{-1}b^{-1}+b^{-2}}{a^{-4}+a^{-2}b^{-2}+b^{-4}}$.

(*b*) Change from radical signs to fractional exponents:

1. $\sqrt{x}+3\sqrt[3]{x^2}-4\sqrt[4]{x^3}$.
2. $8-2\sqrt{y^3}\cdot 5x\sqrt{x^{-a}}$.
3. $\sqrt[5]{a^2b^3}-\sqrt[6]{a^4b^5}-\sqrt[7]{a^6b^7}$.
4. $a\sqrt[3]{x^5}+(\sqrt{x})^3-\sqrt[5]{(x^2y^3)^2}$.
5. $\sqrt[3]{(xy^2z^3)^4}$.
6. $\sqrt[n]{a^mb^nc^rd^s}$.
7. $\sqrt[4]{x^3\sqrt[3]{x^{-4}}}$.
8. $\sqrt{x^{-3}}\div\sqrt[3]{x^{-5}}$.
9. $(\sqrt[4]{b^{-\frac{4}{3}}})^{\frac{2}{3}}$.
10. $\sqrt[3]{(a^{-2}b^{-3})^{-\frac{3}{4}}}$.

(*c*) Change from fractional exponents to radical signs:

1. $x^{\frac{2}{3}}+y^{\frac{3}{5}}$.
2. $7x^{-\frac{5}{3}}y^{-\frac{4}{5}}$.
3. $2\left(\dfrac{a^{-\frac{1}{3}}}{2}\right)^{\frac{1}{2}}$.
4. $3x^{\frac{3m}{2}}y^{\frac{2t}{s}}$.
5. $5a^{\frac{2}{n}}b^{-\frac{3}{n}}c^{-\frac{s}{t}}$.
6. $\{(xy^{-1m})\}^{\frac{1}{s}}$.
7. $\{(a^{-2}b^{-3})^{-\frac{3}{4}}\}^{\frac{1}{3}}$.
8. $a^{-\frac{3}{2}}b^{\frac{2}{3}}\div x^{-\frac{1}{2}}y^{-\frac{7}{2}}$.
9. $x^{-\frac{3}{2}}y(xy^{-2})^{-\frac{1}{2}}$.
10. $(16x^2+5y^2)^{\frac{5}{3}}$.
11. $(a^mb^n)^{\frac{1}{n}}\div(a^nb^m)^{\frac{1}{m}}$.
12. $(a^{-m}b^{-n})^{-\frac{1}{n}}\cdot(a^{-n}b^{-m})^{-\frac{1}{m}}$.

(*d*) Express with positive indices and simplify:

1. $2\,a^{-2}b^{-3}$.
2. $a^{-\frac{7}{5}}b^{-\frac{8}{5}}$.
3. $(x^{\frac{1}{2}}y^2+xy^{\frac{3}{2}})^{-2}$.
4. $3\,ab^{-2}c^{-3}$.
5. $2^{-1}a^{-\frac{1}{2}}cx^{-\frac{3}{2}}$.
6. $\sqrt[5]{a^{-3}} \div \sqrt[6]{x^{-5}}$.
7. $\sqrt[n]{a^{-1}b^{-2}c^{-3}}$.
8. $(-\,x^{-1}b^{-2}a^{-m})^{-\frac{1}{3}}$.
9. $z^3x^{-2}y^{-3} + 3\,x^{-1}z^{-1}y$.
10. $\dfrac{x^{-1}-y^{-2}-z^{-3}}{x^{-4}+y^{-5}-z^{-6}}$.
11. $\dfrac{2\,x^{-2}-3\,y^{-\frac{1}{2}}+4\,z^{-\frac{3}{2}}}{ax^{-1}+by^{-\frac{3}{2}}-cz^{-2}}$.

(*e*) Find the value of each expression:

When $x = y$.

1. $(x-y)^a + a^{x-y} = ?$

When $4\,x = 5$ and $y = 1$.

2. $\{\frac{1}{3}(x^{\frac{3}{2}} + x^{-\frac{1}{2}}y^{-1})\}^{\frac{2}{3}} - \sqrt{\frac{1}{3}\{1 + x^{-\frac{3}{2}} - (1 + xy^{-1})^{-\frac{1}{2}}\}} = ?$

When $x^{-1} = (a-b)(a-c)$, $y^{-1} = (b-c)(a-b)$, and $z^{-1} = (a-c)(b-c)$.

3. $x - y - z = ?$
4. $(\frac{1}{2})^{-8} \cdot 8^{-\frac{4}{3}} \cdot 5^0 \cdot 16^{-\frac{1}{4}} \div (8^{2} \cdot 3)^{-1} = ?$

When $x = 16$ and $y = 81$.

5. $(x^{-\frac{3}{4}} - y^{-\frac{3}{4}}) \div (x^{-\frac{1}{4}} - y^{-\frac{1}{4}}) = ?$

When $x = 10$, $y = 100$, $z = 50$.

6. $x^4y^3z^2 = ?$
7. Prove $x^2y^3z^4 = 5^{16} \times 2^{12}$.

EXAMPLES

ADDITION AND SUBTRACTION

1. $(a^0)^n + (a^n)^0 - (a^0)^0 - a^0b^0 + 1$.
2. $a^0 - 27^{-\frac{2}{3}} + 9^{-\frac{3}{2}} - (a^2 - b^2)^0$.
3. $81^{\frac{3}{4}} + (\frac{1}{32})^{-\frac{2}{5}} + \sqrt[6]{64} - (\frac{25}{49})^{-\frac{1}{2}}$.
4. $25^{-\frac{3}{2}} - 49^{-\frac{1}{2}} + (5\frac{1}{16})^{\frac{3}{4}} + (\frac{25}{16})^{-\frac{3}{2}}$.
5. $(-\,8)^{+\frac{5}{3}} + 125^{-\frac{2}{3}} - (\frac{32}{243})^{-\frac{1}{5}} + (\frac{1}{32})^{-\frac{2}{5}}$.

6. $25^{\frac{3}{2}} - 16^{\frac{3}{4}} - \dfrac{2}{8^{-\frac{2}{3}}} + \dfrac{\sqrt[5]{2}}{4^{-\frac{2}{5}}}.$

7. $\frac{1}{2}(\frac{3}{4})^{\frac{1}{2}} + \frac{1}{2}\sqrt{\frac{3}{9}} - \frac{1}{2}(\frac{3}{25})^{\frac{1}{2}}.$

8. $2\frac{2}{3}\sqrt{\frac{5}{4}} + 5(\frac{16}{5})^{-\frac{1}{2}} - 3(\frac{5}{16})^{\frac{1}{2}}.$

9. $(a^2b^3)^{\frac{1}{5}} + (a^2b^3)^{-\frac{1}{5}} - (a^3b^2)^{-\frac{3}{2}}.$

10. $a(x-y)^{\frac{1}{2}} - b\sqrt{x-y} + c(x^2 - 2\,xy + y^2)^{\frac{1}{4}}.$

MULTIPLICATION

1. $m^{\frac{1}{2}} \cdot m^{\frac{1}{3}}.$

2. $a^{\frac{1}{3}} \cdot a^{-\frac{1}{2}} \cdot a^{\frac{1}{4}}.$

3. $x^{\frac{2}{3}}y^{\frac{1}{2}}z^{-\frac{1}{3}} \cdot x^{-\frac{2}{3}}y^{-2}z^{-\frac{1}{3}}.$

4. $\sqrt{a} \cdot \sqrt{a} \cdot a^{\frac{1}{2}}.$

5. $a^{-\frac{1}{3}}b^{\frac{4}{5}}\sqrt{c^2} \cdot abc.$

6. $(m^{\frac{4}{5}})^{\frac{1}{2}} \cdot (m^{-3})^{-\frac{3}{2}}.$

7. $3\sqrt{xy^{-2}} \cdot 2\,x^{-1}y^{-2}z^{-3}.$

8. $(-\frac{27}{64})^{-\frac{2}{3}} \cdot (-\frac{243}{32})^{-\frac{2}{5}} \cdot 2^{-2}.$

9. $(a^{\frac{2}{3}} + y^{\frac{2}{3}})(a^{\frac{2}{3}} - y^{\frac{2}{3}}).$

10. $(2^{\frac{1}{2}}x^{\frac{1}{2}} + 3^{\frac{1}{2}}y^{\frac{1}{2}})(2^{\frac{1}{2}}x^{\frac{1}{2}} - 3^{\frac{1}{2}}y^{\frac{1}{2}}).$

11. $(x^{(m-1)n} + y^{(n-1)m})(x^n + y^m).$

12. $x^{n-1}y^{n+1} \cdot x^{n+1}y^{1-n}.$

13. $(m^{\frac{2}{3}} + m^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})(m^{\frac{1}{3}} - y^{\frac{1}{3}}).$

14. $(3\,a^{m-1} + 2\,b^{n-2})(2\,a^{1-m} - 3\,b^2).$

15. $(x^n + x^{\frac{n}{2}} + 1)(x^{-n} + x^{-\frac{n}{2}} + 1).$

16. $(2^{a+4} - 2 \times 2^a)(2^{-n-2} \times 4^{-1}).$

17. $(\sqrt{x} - 1 + \sqrt{2})(\sqrt{x} - 1 - \sqrt{2}).$

18. $(a^{-2}x^{-3})^{-2} \cdot (a^{-3}x^{-4})^{-2} \cdot \sqrt{ax} \cdot a^{-5}x^{-7}.$

19. $(m^{\frac{1}{2}} + m^{\frac{1}{4}}n^{\frac{1}{4}} + n^{\frac{1}{2}})(m^{\frac{1}{2}} - m^{\frac{1}{4}}n^{\frac{1}{4}} + n^{\frac{1}{2}}).$

20. $(a^{\frac{9}{4}} + a^{\frac{3}{2}}y^{-\frac{3}{4}} + a^{\frac{3}{4}}y^{-\frac{3}{2}} + y^{-\frac{9}{4}})(a^{\frac{3}{4}} - y^{-\frac{3}{4}}).$

21. $(3\,a^{-2}b^{-3} + 2\,a^{\frac{1}{2}}b^{\frac{3}{4}})(3\,a^{-2}b^{-3} - 2\,a^{\frac{1}{2}}b^{\frac{3}{4}}).$

22. $(x^c - 3\,x^{-c} - 5\,x^0)(x^{-c} + 3\,x^c + 5\,x^0).$

23. $(a^{-\frac{1}{2}}\sqrt{b} + 2 - \sqrt{ab^{-1}})(\sqrt{a^{-1}} - \sqrt{b^{-1}}).$

24. $(2^{-1} + 3^{-2}x^{-2})(2^{-2} - 3^{-2} \cdot x^{-2} + 3^{-4}x^{-4}).$

25. $(\frac{1}{4}\,a^{-1}b^2 + 1 + \frac{1}{2}\,a^{-\frac{1}{2}}b)(\frac{1}{8}\,a^{-\frac{3}{2}}b^3 + 1 - \frac{1}{2}\,a^{-\frac{1}{2}}b).$

DIVISION

1. $\left(\dfrac{x^m}{y^n}\right)^{m-n} \div \left(\dfrac{y^{m+n}}{x^m}\right)^n.$

2. $(5\frac{5}{4})^{\frac{3}{2}} \div (\frac{3}{7})^{-2}.$

3. $\sqrt{xz^{-3}} \div x^{-1}z^{-\frac{1}{2}}.$

4. $a\sqrt{b^{-2}} \div b^{-2}\sqrt[3]{b^{-6}}.$

5. $(2\,m^2n^{-\frac{1}{2}})^{-2} \div 4(mn^{\frac{1}{2}})^4.$

6. $5\,a^{\frac{1}{2}}b^2c^{-3} \div 3^{-1}\,a^{-\frac{2}{3}}b^{-1}c^4.$

7. $\sqrt{a^{-4}b^6} \cdot a^{\frac{1}{2}}\sqrt{b^{-3}} \div a^2b^{-3}.$

8. $\dfrac{x^{\frac{1}{3}}y^{\frac{1}{2}}}{z^{-\frac{2}{3}}} \div \dfrac{y^{-1}z^2}{x^{\frac{1}{2}}} \cdot \dfrac{z^3}{\sqrt{xy^{-\frac{3}{2}}}}.$

9. $\dfrac{a^{\frac{1}{2}}b^{-\frac{2}{3}}}{c^{-1}d^2} \div \dfrac{a^{-\frac{1}{3}}\sqrt[3]{b}}{c^{-1}\sqrt{d}}.$

10. $a^{\frac{1}{3}}\sqrt{b^{-\frac{1}{2}}} \div b^{\frac{3}{4}}\sqrt[3]{a^{-1}}.$

11. $(a^x - b^y) \div (a^{\frac{x}{5}} - b^{\frac{y}{5}}).$

12. $(a^2 - b^{-2}) \div (a^{\frac{1}{2}} - b^{-\frac{1}{2}}).$

13. $(a^{6x} - b^{-6y}) \div (a^x - b^{-y}).$

14. $(16\,a - b^2) \div (2\,a^{\frac{1}{4}} - b^{\frac{1}{2}}).$

15. $(x^{-3} - 64\,y^2) \div (x^{-\frac{1}{2}} - 2\,y^{\frac{1}{3}}).$

16. $\dfrac{a^{-1}b^{-2}}{ab^{-3}} \cdot \dfrac{a^2b^{-1}}{b^4} \div \left(\dfrac{a^{-2} - b^{-1}}{a^2}\right)^2.$

17. $\left(\sqrt{x^3\sqrt{y^{-2}}}\right)^6 \div \left(\sqrt[3]{x^2\sqrt{y^{-3}}}\right)^6.$

18. $(x^p + x^{\frac{p}{2}}y^{\frac{p}{2}} + y^p) \div (x^{\frac{p}{2}} - x^{\frac{p}{4}}y^{\frac{p}{4}} + y^{\frac{p}{2}}).$

19. $(x + y + 3\,x^{\frac{1}{3}}y^{\frac{1}{3}} - 1) \div (x^{\frac{1}{3}} + y^{\frac{1}{3}} - 1).$

20. $(x + y - 3\,x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + z) \div (x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}).$

21. $(16\,x^{-3} + 6\,x^{-2} + 5\,x^{-1} - 6) \div (2\,x^{-1} - 1).$

22. $(5\,x^{\frac{2}{3}} - 6\,x^{\frac{1}{3}} - 4\,x^{-\frac{1}{3}} - 24\,x^{-\frac{2}{3}}) \div (x^{\frac{1}{6}} - 2\,x^{-\frac{1}{6}}).$

23. $(x^{-3r} - x^{-2r}y^{-s} + \frac{1}{3}\,x^{-r}y^{-2s} - \frac{1}{27}\,y^{-3s}) \div (x^{-r} - \frac{1}{3}\,y^{-s}).$

24. $(x^{\frac{3}{p}} - (a + b + c)\,x^{\frac{2}{p}} + (ab + bc + ca)\,x^{\frac{1}{p}} - abc)$
$\div (x^{\frac{1}{p}} - c).$

25. $(a^{\frac{3}{4}} - \frac{3}{5}\,a^{\frac{7}{10}} - \frac{2}{3}\,a^{\frac{7}{12}} + \frac{2}{5}\,a^{\frac{8}{15}} + \frac{1}{2}\,a^{\frac{1}{2}} - \frac{3}{10}\,a^{\frac{9}{20}})$
$\div (a^{\frac{1}{4}} - \frac{3}{5}\,a^{\frac{1}{5}}).$

26. $\dfrac{6\,xy}{m^{\frac{1}{2}} + n^{\frac{1}{2}}} \div \left[\dfrac{3(m^{\frac{1}{2}} - n^{\frac{1}{2}})x^{\frac{1}{2}}}{7(r^{\frac{1}{2}} + s^{\frac{1}{2}})} \div \left(\dfrac{4(r^{\frac{1}{2}} + s^{\frac{1}{2}})}{21\,x^{\frac{1}{2}}y} \div \dfrac{r - s}{4(m - n)}\right)\right].$

FACTORING

1. $91\,a^{\frac{2}{3}}m^{-\frac{3}{5}}z^3$.
2. $a^{-3}-b^{-3}$.
3. $a^{-3}+b^{-3}$.
4. $a^{-6}-b^{-6}$.
5. $a^{-6}+b^6$.
6. $x+2\,x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.
7. $9\,x^{-4}-25\,y^{-4}$.
8. $x^{\frac{4}{3}}+4\,y^{-\frac{4}{3}}$.
9. $x^2-1+x^{\frac{1}{2}}-x^{\frac{3}{2}}$.
10. $8\,s^{-2}-6\,s^{-1}t^{-1}-5\,t^{-2}$.
11. $4\,a^{-2}-12\,a^{-1}b^{-1}+9\,b^{-2}$.
12. $9\,a^{-4}+12\,a^{-2}x^{-1}+4\,x^{-2}$.
13. $9\,a^{\frac{3}{2}}+3\,a^{\frac{3}{4}}b^{\frac{1}{2}}-2\,b$.
14. $3\,x^{\frac{3}{2}}-5\,x-12\,x^{\frac{1}{2}}+20$.
15. $16\,x-8\sqrt{3\,x}+3$.
16. $4\,x^{\frac{3}{2}}-7\,x-16\,x^{\frac{1}{2}}+28$.
17. $x^{\frac{4}{3}}-3\,x^{\frac{2}{3}}-4\sqrt{x}-3$.
18. $16\,x-\sqrt{y}-6\sqrt{x\sqrt{y}}$.
19. $x+a+3\,a^{\frac{1}{3}}b^{\frac{1}{3}}x^{\frac{1}{3}}-b$.
20. $x-2\sqrt{xy}+y-\sqrt{x}+\sqrt{y}$.
21. $5\sqrt{x}+5\sqrt{y}+5\sqrt{5\sqrt{x}+5\sqrt{y}}-50$.
22. $ax^{-1}-2\,a^{\frac{1}{2}}x^{-\frac{1}{2}}+3-2\,a^{-\frac{1}{2}}x^{\frac{1}{2}}+a^{-1}x$.
23. $x^2-9\,xy^2-8\,xy\sqrt{x^2-9\,xy^2}+16\,x^2y^2$.
24. $x^{-4}-x^{-3}y^{-1}+\frac{9}{4}\,x^{-2}y^{-2}-x^{-1}y^{-3}+y^{-4}$.
25. $a^{-2}-2\,a^{-1}c^{-1}+c^{-2}-b^{-2}-2\,b^{-1}d^{-1}-d^{-2}$.
26. $-6-(4\,x^2-9\,x+11)^{\frac{1}{2}}-(9\,x-4\,x^2-11)$.

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27. $\begin{cases} 3\,a^{\frac{3}{2}}x^3+5\,ax^2-a^{\frac{1}{2}}x+2. \\ 4\,a^2x^4+9\,a^{\frac{3}{2}}x^3+2\,ax^2-2\,a^{\frac{1}{2}}x-4. \end{cases}$
28. $ax^3-a^{\frac{1}{3}}x,\ ax^3-1,\ ax^3+1$.
29. $2\,a^{\frac{4}{3}}-a^{\frac{2}{3}}b^{\frac{3}{2}}-3\,b^3,\ 10\,a^{\frac{4}{3}}-15\,a^{\frac{2}{3}}b^{\frac{3}{2}}$.
30. $24\,a^3+112\,a^2-94\,a+18,\ 6\,a^3+25\,a^2-21\,a+4$.
31. $x^8+x^4y^4+y^8,\ x^4-x^2y^2+y^4,\ x^2+xy\sqrt{3}+y^2$.
32. $x^4+y^4,\ x^2-xy\sqrt{2}+y^2,\ x^8-y^8,\ x^4-y^4$.
33. $16\,x-8\sqrt{3\,x}+3,\ 243+324\sqrt{3\,x},\ 16\,x-3$.

INVOLUTION

1. $(3a^{-2}+2b^{-\frac{1}{3}})^2$.
2. $(\frac{2}{3}x^{-1}-\frac{3}{2}x^{-2})^2$.
3. $(3a^{-\frac{1}{m}}-2b^{-\frac{m}{n}})^3$.
4. $\left(\frac{a^{-2}}{x^{-2}}+\frac{x^{-2}}{a^{-2}}\right)^2$.
5. $(2x^{-1}-3y^{-\frac{3}{4}})^2$.
6. $(x^{-1}y-2y^2)^3$.
7. $(2a^{\frac{m}{n}}-3b^{-\frac{1}{n}})^3$.
8. $(m^{-p}-n^{-q})^3$.
9. $(a^{-1}-2b^{-2}-3c^{\frac{1}{2}}+4c^{-\frac{3}{4}})^2$.
10. $(5x^{\frac{1}{2}}-y^{\frac{1}{3}})^3$.
11. $(2a^2m^{\frac{4}{3}}z^{-2})^5$.
12. $(5x^ay^{-b})^{\frac{m}{n}}$.
13. $(16a^{15}x^{-6})^{\frac{2}{3}}$.
14. $(-\frac{2}{3}a^{\frac{1}{2}}m^{-\frac{1}{3}}x)^{-\frac{3}{4}}$.
15. $(64a^{18}b^{-24})^{-\frac{5}{6}}$.
16. $(2a-3a^{-\frac{2}{3}})^3$.
17. $(2x^{\frac{3}{2}}-ax^{-\frac{2}{3}})^4$.
18. $(x^{-\frac{1}{2}}y^{-\frac{3}{2}}+\frac{3}{4}x^{\frac{2}{3}}y^{-\frac{2}{3}})^4$.

385. As the **Binomial Theorem** is true for all values of n, its uses may be extended to the cases of fractional and negative exponents, provided $a > b$.

From Art. **356**,

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \cdots$$

$$\cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}a^{n-r}b^r + \cdots + nab^{n-1} + b^n.$$

That a must be greater than b in the expression $(a+b)^n$ may be seen from the following example:

Let $\qquad a = 1,\ \ b = -x,$ and $n = -1;$

then $\qquad (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots.$

Since x is unrestricted in value, substitute 2 for x; then $(1-2)^{-1} = 1 + 2 + 4 + 8 + \cdots$, which is evidently untrue.

Hence $(1+x)^n \neq 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 + \cdots$ for values of $x > 1$. Therefore in the expansion of $(a+b)^n$, a must be greater than b.

By applying the laws of the Binomial Theorem as given in Art. **354**, or by substituting in the formula, any binomial can be affected with any exponent.

Model Solutions

1. $$(1-\tfrac{2}{3}x^2)^{\frac{2}{3}} = 1^{\frac{2}{3}} + \tfrac{2}{3}\cdot 1^{-\frac{1}{3}}\cdot(-\tfrac{2}{3}x^2) - \tfrac{1}{9}\cdot 1^{-\frac{4}{3}}\cdot(-\tfrac{2}{3}x^2)^2 + \tfrac{4}{81}\cdot 1^{-\frac{7}{3}}\cdot(-\tfrac{2}{3}x^2)^3 - \cdots$$
$$= 1 - \tfrac{4}{9}x^2 - \tfrac{4}{81}x^4 - \frac{2^5}{3^7}x^6 - \cdots.$$

2. $$\frac{1}{(a^4-2^{-1}b^{-\frac{1}{2}})^{\frac{1}{2}}} = (a^4 - 2^{-1}b^{-\frac{1}{2}})^{-\frac{1}{2}} = (a^4)^{-\frac{1}{2}} - \tfrac{1}{2}(a^4)^{-\frac{3}{2}}(-2^{-1}b^{-\frac{1}{2}})$$
$$+ \tfrac{3}{8}(a^4)^{-\frac{5}{2}}(-2^{-1}b^{-\frac{1}{2}})^2 - \tfrac{5}{16}(a^4)^{-\frac{7}{2}}(-2^{-1}b^{-\frac{1}{2}})^3 + \frac{35}{2^7}(a^4)^{-\frac{9}{2}}(-2^{-1}b^{-\frac{1}{2}})^4 - \cdots$$
$$= a^{-2} + \tfrac{1}{4}a^{-6}b^{-\frac{1}{2}} + \frac{3}{2^5}a^{-10}b^{-1} + \frac{5}{2^7}a^{-14}b^{-\frac{3}{2}} + \frac{35}{2^{11}}a^{-18}b^{-2} + \cdots$$
$$= \frac{1}{a^2} + \frac{1}{4a^6b^{\frac{1}{2}}} + \frac{3}{2^5a^{10}b} + \frac{5}{2^7a^{14}b^{\frac{3}{2}}} + \frac{13}{2^{11}a^{18}b^2} + \cdots.$$

3. $$\sqrt[3]{6} = \sqrt[3]{8-2} = (8-2)^{\frac{1}{3}} = 8^{\frac{1}{3}} + \tfrac{1}{3}\cdot 8^{-\frac{2}{3}}\cdot(-2) - \tfrac{1}{9}\cdot 8^{-\frac{5}{3}}\cdot(-2)^2$$
$$+ \tfrac{5}{81}\cdot 8^{-\frac{8}{3}}\cdot(-2)^3 - \tfrac{10}{243}\cdot 8^{-\frac{11}{3}}\cdot(-2)^4 + \cdots$$
$$= 2 - \frac{1}{6} - \frac{1}{72} - \frac{5}{3^4\cdot 2^5} - \frac{5}{3^5\cdot 2^6} - \cdots,$$

4. Find the 5th term of $(2x^2 - \frac{1}{2}y^{\frac{1}{3}}z^{-\frac{1}{2}})^{-\frac{1}{4}}$.

General term $= \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}a^{n-r}b^r.$

$n = -\frac{1}{4}$,	$= \dfrac{-\frac{1}{4}(-\frac{5}{4})(-\frac{9}{4})(-\frac{13}{4})}{1\cdot 2\cdot 3\cdot 4}(2x^2)^{-\frac{17}{4}}(-\frac{1}{2}y^{\frac{1}{3}}z^{-\frac{1}{2}})^4$
$r = 4$,	
$a = 2x^2$,	$= +\dfrac{5\cdot 9\cdot 13}{2^{11}\cdot 3}\cdot\dfrac{1}{2^{\frac{17}{4}}x^{\frac{17}{2}}}\cdot\dfrac{y^{\frac{4}{3}}}{2^4z^2}$
$b = -\frac{1}{2}y^{\frac{1}{3}}z^{-\frac{1}{2}}$,	
$n - r + 1 = -\frac{13}{4}$.	$= \dfrac{195\,y^{\frac{4}{3}}}{2^{19}\cdot\sqrt[4]{2}\cdot x^{\frac{17}{2}}\cdot z^2}.$

Expand to five terms when not terminating:

1. $(x+y)^{\frac{1}{2}}$.
2. $(a-b)^{\frac{1}{3}}$.
3. $(x^{\frac{1}{3}}-y^{-\frac{1}{5}})^{-5}$.
4. $\dfrac{1}{(1-3x^{\frac{1}{2}})^{-4}}$.
5. $\dfrac{1}{(2x^{-\frac{1}{2}}-3y^{-\frac{3}{5}})^{-5}}$.
6. $\dfrac{1}{(2x^{-1}-3y)^{-\frac{1}{2}}}$.
7. $(c-d)^{\frac{3}{4}}$.
8. $(x-2y^2)^{-\frac{1}{2}}$.
9. $\left(\dfrac{a^{-1}}{a^{\frac{1}{3}}-2b^{-2}}\right)^{-6}$.
10. $\left(\dfrac{c^{-2}}{x^{-2}-3x^{-\frac{1}{7}}}\right)^{-7}$.
11. $\left(\dfrac{x^{-2}+y^{-2}}{x^{-6}+y^{-6}}\right)^{-3}$.
12. $(1-\frac{1}{5}x)^{n}$.
13. $\sqrt[3]{1-x^2}$.
14. $\sqrt[4]{1-3x^{-1}}$.
15. $(1-x^2)^{\frac{5}{2}}$.
16. $(1-\frac{2}{3}x)^{-\frac{2}{3}}$.
17. $(a^2x-x^2)^{\frac{1}{2}}$.
18. $\dfrac{a^2}{(a+b)^2}$.
19. $(a+x^{-2})^{-\frac{2}{3}}$.

20. Find the 4th term of $(x^{\frac{1}{2}}+y^3)^6$.
21. Find the 5th term of $(2x^{-2}-y^{-\frac{1}{2}})^8$.
22. Find the 5th term of $(2x-3y)^{\frac{1}{2}}$.
23. Find the 4th term of $(x^{-1}-y^{-1})^{\frac{1}{4}}$.
24. Find the 8th term of $(2ab-3cd)^{-1}$.
25. Find the middle term of $(a^{-2}-x^{-2})^8$.
26. Find the 6th term of $(3^{-\frac{1}{7}}-\frac{1}{2}x)^{-7}$.
27. Find the 7th term of $(a^{-100}-\frac{1}{2}x^{-\frac{1}{6}})^6$.
28. Find the two middle terms of $(x^{-\frac{1}{2}}-y^{\frac{1}{2}})^{15}$.

EVOLUTION

Review and use the Laws of Evolution in Chapter XII.:

1. $\sqrt[6]{64a^{-5}x^{-12}}$.
2. $\sqrt[5]{2^{-5}a^0x^{-\frac{10}{7}}y^{-\frac{2}{5}}}$.
3. $\sqrt[4]{(a^2b^{-\frac{1}{2}}c^{\frac{2}{3}})^{-1}}$.
4. $\sqrt[3]{\{-(a^3)^{\frac{1}{2}}\}^{-1}}\cdot\sqrt[4]{\{-(-a)^{-3}\}^2}$.
5. $\sqrt[4]{a^{-\frac{1}{3}}b^2c\sqrt{x^{-1}}}$.
6. $\sqrt{(\{(x^4)^{\frac{1}{4}}y^6\}^{\frac{1}{2}}z^6)^{-1}}$.
7. $\sqrt{(a^{\frac{1}{2}}-2a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}})}$.
8. $\sqrt{(4x^{-4}+12x^{-3}+9x^{-2})}$.

9. $\sqrt{(x^3 + 2x^{\frac{5}{2}} - 3x^2 - 4x^{\frac{3}{2}} + 4x)}$.

10. $\sqrt[3]{(x^{\frac{3}{5}} - a^{\frac{1}{10}}x^{\frac{2}{5}} + \frac{1}{3}a^{\frac{1}{5}}x^{\frac{1}{5}} - \frac{1}{27}a^{\frac{3}{10}})}$.

11. $\sqrt{(a^{\frac{4}{3}} + 4ay^{\frac{1}{5}} + 10a^{\frac{2}{3}}y^{\frac{2}{5}} + 12a^{\frac{1}{3}}y^{\frac{3}{5}} + 9y^{\frac{4}{5}})}$.

12. $\sqrt{(4x^{\frac{4}{3}} - 12xy^{\frac{1}{2}} - 7x^{\frac{2}{3}}y + 24x^{\frac{1}{3}}y^{\frac{3}{2}} + 16y^2)}$.

13. $\sqrt{(16a^{\frac{2}{3}} + 24a^{\frac{1}{3}}\sqrt{x} - 7x - 12a^{-\frac{1}{3}}x^{\frac{3}{2}} + 4a^{-\frac{2}{3}}x^2)}$.

14. $\sqrt{(xy^{\frac{1}{2}} + 4x^{\frac{1}{2}}y^{\frac{7}{12}} + 4x^{-\frac{1}{2}}y^{\frac{1}{3}} + x^{-1} + 2y^{\frac{1}{4}} + 4y^{\frac{2}{3}})}$.

15. $\sqrt{(1 + 4x^{-\frac{1}{3}} - 2x^{-\frac{2}{3}} - 4x^{-1} + 25x^{-\frac{4}{3}} - 24x^{-\frac{5}{3}} + 16x^{-2})}$.

MISCELLANEOUS EXAMPLES

1. $\left(\dfrac{a^0b^{-2}}{c^{-3}}\right)^6$.

2. $(1 + x^{-1})^n$.

3. $\dfrac{\sqrt{50\,a^6b^4}}{\sqrt{2\,a^2b^2}}$.

4. $(x^{\frac{2}{3}} \cdot x^{\frac{4}{5}})^{\frac{5}{11}}$.

5. $\dfrac{a}{b}\sqrt[3]{\dfrac{a^{-2}b^0}{ab^{-3}}}$.

6. $(3^{\frac{1}{3}} - 3^{-\frac{1}{3}})^3$.

7. $\left(\dfrac{a^0b^{-1}}{a^{-2}b^2}\right)^5 \div \left(\dfrac{a^{-2}b}{ab^{-4}}\right)^{-3}$.

8. $\{(2a - 3b)\sqrt{2x - y}\}^2$.

9. $\dfrac{\sqrt{a^{-4}b^6} \cdot \sqrt{a\sqrt{b^{-3}}}}{a^2b^{-3}}$.

10. $\sqrt{a^{-\frac{5}{3}}\sqrt{b^3}} \div \sqrt{a^{\frac{1}{2}}\sqrt[3]{b^{-\frac{3}{4}}}}$.

11. $\{(x^{m+n})^{m-n} + (x^n)^{n+m}\} \div (x^m)^{m+n}$.

12. $(xyz^2)^{\frac{1}{3}} \cdot (x^{\frac{4}{3}}y^{-1}z^{\frac{1}{3}}) \cdot (x^{-\frac{2}{3}}y^{\frac{2}{3}}z^{-1})$.

13. Expand to four terms $\dfrac{1}{\sqrt{1 + x^2}}$.

14. $\left(1 - \dfrac{1 - x^{\frac{1}{2}}}{1 + x^{\frac{1}{2}}}\right)^3$.

15. $\dfrac{1 - x^{-2} - y^{-2}}{1 - x^{-3}y^{-2} + x^{-2}}$.

16. $(2x\sqrt{y} - 3y\sqrt[3]{x^{-2}})^3$.

17. $\dfrac{2^{n+1} \cdot 2^{2n}}{(2^n)^{n+1}} \div \dfrac{4^{n+1}}{(2^{n-1})^{n+1}}$.

18. $(5a^{\frac{1}{2}}b^2c^{-3})(3a^{\frac{2}{3}}b^{-1}c^4)$.

19. $(x^{\frac{3n}{2}} - y^{\frac{3n}{2}}) \div (x^{\frac{n}{2}} - y^{\frac{n}{2}})$.

20. $\left\{\dfrac{\sqrt[3]{a}}{\sqrt[4]{b}} \cdot \left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right\}^4$.

21. Factor $a^{\frac{1}{2}}b^{2m}c^{-2} + a^{-\frac{1}{2}}b^{-2m}c^2 - 2$.

22. $(x^2y^{-\frac{4}{3}} - 2 + x^{-2}y^{\frac{4}{3}}) \div (x^{\frac{1}{2}}y^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{1}{3}})$.

23. $\sqrt{(x^3y^{-\frac{2}{3}} - 4x^{\frac{3}{2}}y^{-\frac{1}{3}} + 6 - 4x^{-\frac{3}{2}}y^{\frac{1}{3}} + x^{-3}y^{\frac{2}{3}})}$.

24. $\sqrt{(4^{-1}a^4 + a^3x^{-1} + a^2x^{-2} - ax - 2 + a^{-2}x^2)}$.

SYNOPSIS FOR REVIEW, CHAPTER XIII

GENERAL THEORY OF EXPONENTS

Exponent, **373**.

Laws
1. The mth power of a quantity multiplied by the nth power of that quantity, **374**.
2. The mth power of a quantity divided by the nth power of that quantity, **374**.
3. The nth power of the mth power of a quantity, **374**.
4. The nth power of the product of two or more factors, **374**.
5. The nth power of the quotient of two factors, **374**.

The Fundamental Index Law, **376**.

Props.
1. Any root of any power of a quantity, **375**. Dem.
2. The nth root of the mth root of any quantity, **383**. Dem.

Probs.
1. To find the meaning of any positive fractional exponent, **377**.

   A positive fractional exponent indicates a power of a root, or a root of a power, **378**.
2. To find the meaning of any negative exponent, **379**.

   A negative exponent of an expression signifies the reciprocal of the expression with the sign of the exponent changed, **380**.
3. To prove the 5 Laws of Exponents when m and n are positive fractions, **384**. Dem.
4. To prove the 5 Laws of Exponents when m and n are negative, **386**. Dem.

Cors.
1. The sign of the exponent of any quantity may be changed by changing the sign belonging to the quantity from $\times$ to $\div$, or $\div$ to $\times$, **381**.
2. A factor may be transferred from either term of a fraction by changing the sign of the exponent of the factor, **382**.
3. Multiplying both terms of a fractional exponent by the same number does not change its value, **385**.

Sample Test Questions

1. State the 5 General Laws of Exponents.
2. State the Fundamental Index Law, and give its demonstration.
3. State the meaning of a positive fractional exponent.
4. State the meaning of a negative exponent.
5. How may the sign of an exponent be changed?
6. How may a multiplier be changed into a divisor, and *vice versa*?
7. How are factors transferred from either term of a fraction?
8. State Prop. 1, and give its demonstration; state Prop. 2, and give its demonstration.
9. State Prob. 3, and give its demonstration; state Prob. 4, and give its demonstration.
10. State the laws for expanding a binomial; state the General Term.

CHAPTER XIV

RADICAL EXPRESSIONS

SECTION I

DEFINITIONS AND REDUCTION

386. Define radical number, surd, root, degree of root, index, rational, irrational, imaginary.

387. Reduction of Radicals is the process of changing the form of a radical number without changing its value.

A radical is in its *simplest form* when the quantity under the radical sign is in the smallest integral form consistent with the degree of the root, as $\sqrt{2}$.

388. Similar Radicals are like roots of like quantities, as $a\sqrt[3]{2b}$, $x\sqrt[3]{2b}$, $m\sqrt[3]{2b}$.

389. PROBLEM 1. **To reduce a radical to its simplest form when one of the factors of the expression under the radical sign is a perfect power corresponding to the index of the root.**

Rule. *Separate the radical into two factors, rational and surd. Extract the root indicated in the rational factor and write the result as a coefficient of the surd factor.*

Dem. See law 2 of Evolution. State the law and give its demonstration.

MODEL SOLUTIONS

1. $\sqrt[3]{108} = \sqrt[3]{27 \cdot 4} = \sqrt[3]{27} \cdot \sqrt[3]{4} = 3\sqrt[3]{4}$. Why?

2. $\sqrt{2x^2 - 4xy + 2y^2} = \sqrt{(x^2 - 2xy + y^2)2} = \sqrt{(x-y)^2} \cdot \sqrt{2} = (x-y)\sqrt{2}$.

390. Cor. A surd fraction may be simplified by multiplying both terms of the fraction by the smallest number which will make the denominator a perfect power corresponding to the degree of the root, extracting the required root of the denominator, and writing the result as a fractional coefficient of the surd factor.

Model Solutions

1. $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{4}} \cdot \sqrt{2} = \frac{1}{2}\sqrt{2}$. Why?

2. $\sqrt[3]{\frac{1}{4}} = \sqrt[3]{\frac{2}{8}} = \sqrt[3]{\frac{1}{8}} \cdot \sqrt[3]{2} = \frac{1}{2}\sqrt[3]{2}$.

3. $\sqrt[5]{\frac{2}{3}} = \sqrt[5]{\frac{2 \cdot 3^4}{3^5}} = \sqrt[5]{\frac{1}{3^5}} \cdot \sqrt[5]{2 \cdot 3^4} = \frac{1}{3}\sqrt[5]{2 \cdot 3^4}$.

4. $\sqrt[n]{\frac{a}{b}} = \sqrt[n]{\frac{ab^{n-1}}{b^n}} = \sqrt[n]{\frac{1}{b^n}} \cdot \sqrt[n]{ab^{n-1}} = \frac{1}{b}\sqrt[n]{ab^{n-1}}$.

5. $3\sqrt[3]{128\, a^5b^6c^8} = 3\sqrt[3]{64\, a^3b^6c^6 \cdot 2\, a^2c^2} = 3\sqrt[3]{64\, a^3b^6c^6} \cdot \sqrt[3]{2\, a^2c^2}$
$= 12\, ab^2c^2\sqrt[3]{2\, a^2c^2}$.

EXAMPLES

Reduce to the simplest form:

1. $\sqrt{12}$.
2. $\sqrt[3]{24}$.
3. $\sqrt[4]{80}$.
4. $\sqrt[5]{96}$.
5. $\sqrt[3]{\frac{2\,a}{3\,b}}$.
6. $\sqrt{\frac{x+y}{x-y}}$.
7. $\sqrt[4]{\frac{3}{8}}$.
8. $\sqrt{\frac{a}{b}}$.
9. $\sqrt[6]{448}$.
10. $\sqrt[3]{\frac{1}{x-a}}$.
11. $\sqrt[4]{80\, x^5y^9z^2}$.
12. $5\sqrt[3]{54\, a^5c^{10}x^4}$.
13. $\sqrt[s]{x^{2s}y^{2s+n}}$.
14. $\sqrt[n]{x^{n+1}y^{2n+1}}$.
15. $\sqrt[t]{x^{t+m}y^{2t+3}}$.
16. $\sqrt[m+n]{x^{2m+2n}y^2}$.
17. $\frac{x-1}{x+1}\sqrt{\frac{x+1}{x-1}}$.
18. $\left\{\sqrt[4]{(-\frac{8}{27})^{-\frac{2}{3}}}\right\}^{\frac{2}{5}}$.
19. $4\sqrt[3]{27\, x^3+81\, x^2y+81\, xy^2+27\, y^3}$.
20. $\sqrt{(125\, a^8-375\, a^7x+375\, a^6x^2-125\, a^5x^3)}$.
21. $\sqrt{(a^3+ab^2+ac^2+2\, a^2b+2\, a^2c+2\, abc)}$.

391. PROBLEM 2. **To introduce a coefficient under the radical sign.**

Rule. *Raise the coefficient to the power indicated by the degree of the radical, and multiply the result by the number under the given radical, placing the radical sign over the product.*

Dem. See Articles **383** and **365**, law 2.

Let the student state the reasons in full.

MODEL SOLUTIONS

1. $3\sqrt{2} = \sqrt{9} \cdot \sqrt{2} = \sqrt{18}$. Why?

2. $(m-n)\sqrt[n]{x} = \sqrt[n]{(m-n)^n x}$. Why?

EXAMPLES

Introduce the coefficient in each of the following:

1. $a\sqrt{a}$.	5. $\frac{2}{3}\sqrt{\frac{3}{2}}$.	9. $\frac{2}{3}xy = \sqrt{?}$.
2. $3a^2\sqrt{b}$.	6. $\frac{5}{3}\sqrt[3]{9\frac{3}{5}}$.	10. $(\frac{2}{3})^{-\frac{1}{2}} = \sqrt[6]{?}$.
3. $(a-b)\sqrt[3]{2}$.	7. $\frac{5}{3}x(\frac{5}{9})^{-\frac{1}{2}}$.	11. $\sqrt[3]{3\sqrt{3}} = \sqrt[6]{?}$.
4. $xm\sqrt[3]{a-x}$.	8. $\frac{3}{4}(\frac{2}{3}a)^{-\frac{1}{3}}$.	12. $(a+b)\sqrt{\frac{a-b}{a+b}}$.

392. PROBLEM 3. **To simplify an expression of the form $\sqrt[mn]{a^m}$.**

Let the student describe the nature of this problem and give a rule for simplifying it.

Dem. The *mn*th root of any quantity equals the *n*th root of the *m*th root of that quantity. See Prop. 2 in the chapter on General Theory of Exponents.

MODEL SOLUTIONS

1. $\sqrt[6]{27} = \sqrt{\sqrt[3]{27}} = \sqrt{3}$. State reasons for each step.

2. $\sqrt[mn]{(a-b)^{ms}} = \sqrt[n]{\sqrt[m]{(a-b)^{ms}}} = \sqrt[n]{(a-b)^s}$. State reasons for each step.

EXAMPLES

Simplify:

1. $\sqrt[6]{49\,a^2}$.
2. $\sqrt[12]{8\,a^6}$.
3. $\sqrt[9]{64\,a^3-192\,a^2b+192\,ab^2-64\,b^3}$.
4. $\sqrt[8]{16}$.
5. $\sqrt[10]{9\,a^{10}}$.
6. $\sqrt[3]{3\sqrt{3}}$.
7. $\sqrt[3]{81\sqrt{3}}$.
8. $\sqrt[15]{32\,a^5b^{10}c^{15}}$.
9. $\sqrt[5]{486\,y\sqrt[3]{4\,y^2}}$.
10. $\sqrt[14]{128\,(ax-b)^7}$.

393. PROBLEM 4. **To reduce radicals to the same degree.**

Rule. *Express the indicated roots by means of fractional exponents, reduce the exponents to forms having the L. C. D., and change back to the radical form.*

Dem.
$$\sqrt[m]{a^b}=a^{\frac{b}{m}}=a^{\frac{bn}{mn}}=\sqrt[mn]{a^{bn}}.\quad \text{Why?}$$
$$\sqrt[n]{x^s}=x^{\frac{s}{n}}=x^{\frac{sm}{mn}}=\sqrt[mn]{x^{sm}}.$$

What is the meaning of a fractional exponent? What is the effect on an expression of multiplying both terms of its fractional exponent by the same number? Why?

MODEL SOLUTIONS

1. Reduce $\sqrt{2\,x^2y^3}$, $\sqrt[3]{5\,ab^4}$, $\sqrt[4]{3\,v^5z^{10}}$ to the same degree.

$$\sqrt{2\,x^2y^3}=(2\,x^2y^3)^{\frac{1}{2}}=(2\,x^2y^3)^{\frac{6}{12}}=\sqrt[12]{(2\,x^2y^3)^6}=\sqrt[12]{2^6x^{12}y^{18}}.$$
$$\sqrt[3]{5\,ab^4}=(5\,ab^4)^{\frac{1}{3}}=(5\,ab^4)^{\frac{4}{12}}=\sqrt[12]{(5\,ab^4)^4}=\sqrt[12]{5^4a^4b^{16}}.$$
$$\sqrt[4]{3\,v^5z^{10}}=(3\,v^5z^{10})^{\frac{1}{4}}=(3\,v^5z^{10})^{\frac{3}{12}}=\sqrt[12]{(3\,v^5z^{10})^3}=\sqrt[12]{3^3v^{15}z^{30}}.$$

2. Reduce $\sqrt[4]{30}$ and $\sqrt[5]{90}$ to the same degree.

$$\sqrt[4]{30}=(30)^{\frac{1}{4}}=(30)^{\frac{5}{20}}=\sqrt[20]{(30)^5}=\sqrt[20]{(2\cdot 3\cdot 5)^5}=\sqrt[20]{2^5\cdot 3^5\cdot 5^5}.$$
$$\sqrt[5]{90}=(90)^{\frac{1}{5}}=(90)^{\frac{4}{20}}=\sqrt[20]{(90)^4}=\sqrt[20]{(2\cdot 3\cdot 3\cdot 5)^4}=\sqrt[20]{2^4\cdot 3^8\cdot 5^4}.$$

EXAMPLES

1. $\sqrt{3}$, $\sqrt[3]{2}$.
2. $\sqrt{2}$, $\sqrt[3]{3}$.
3. $\sqrt[3]{3}$, $\sqrt[4]{5}$.
4. $\sqrt[6]{3}$, $\sqrt[8]{2}$.
5. $\sqrt[n]{a}$, $\sqrt[m]{b}$.
6. $\sqrt[s]{a^r}$, $\sqrt[t]{a^v}$.
7. a, $\sqrt{b}$, $\sqrt[3]{c}$.
8. $\sqrt[5]{2}$, $\sqrt[3]{3}$, $\sqrt[10]{4}$.
9. $\sqrt[2m]{a^n}$, $\sqrt[3n]{b^m}$.
10. 2, 3, $\sqrt{2}$, $\sqrt[3]{3}$.
11. $\sqrt[5r]{a-b}$, $\sqrt[4Q]{x}$.
12. $\sqrt[m-n]{a^{m+n}}$, $\sqrt[m+n]{a^{m-n}}$.

SECTION II

ADDITION AND SUBTRACTION

394. PROBLEM 5. **To add or subtract radicals.**

Rule. *If the terms are similar, or can be made so, combine their coefficients and affix the common base. If the terms cannot be made similar, write them in succession with their proper signs.*

Dem. The same as for ordinary addition and subtraction.

Let the student review the demonstrations for addition and subtraction of simple numbers.

MODEL SOLUTIONS

1. $\sqrt[3]{54} + \sqrt[3]{16} = 3\sqrt[3]{2} + 2\sqrt[3]{2} = 5\sqrt[3]{2}$. Why?

2. $\sqrt[3]{1372\,a^4x^5} - \sqrt[3]{500\,a^4x^5} = 7\,ax\sqrt[3]{4\,ax^2} - 5\,ax\sqrt[3]{4\,ax^2} = 2\,ax\sqrt[3]{4\,ax^2}$.

3. $\sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}} + \frac{1}{2}\sqrt{3} = \frac{1}{2}\sqrt{3} + \frac{1}{3}\sqrt{3} + \frac{1}{2}\sqrt{3} = \frac{4}{3}\sqrt{3}$.

4. $\sqrt{2} - \sqrt{3} + \sqrt[3]{3} + \sqrt[n]{5} = \sqrt{2} - \sqrt{3} + \sqrt[3]{3} + \sqrt[n]{5}$.

EXAMPLES

1. $2\,abc\sqrt{20} + 3\,a\sqrt{5\,b^2c^2}$.
2. $\sqrt{x^4y} + \sqrt{z^4y} - \sqrt{4\,x^2yz^2}$.
3. $3\sqrt{\frac{2}{3}} + 2\sqrt{\frac{3}{2}} - 4\sqrt{\frac{1}{6}} + 3\sqrt{6}$.
4. $\sqrt{75} - \sqrt{48} - \sqrt{147} + \sqrt{300}$.
5. $\sqrt{x+y} - \sqrt{xy^2+y^3} + \sqrt{(x+y)^3}$.
6. $\sqrt{18\,a^5b^2} + \sqrt{50\,a^3b^2} + \sqrt{18\,a^3bx^2}$.
7. $2\sqrt[3]{40} + 3\sqrt[3]{108} + \sqrt[3]{500} - \sqrt[3]{320}$.
8. $\frac{3}{4}\sqrt{\frac{2}{3}} - (-\frac{2}{5}\sqrt{\frac{1}{6}})$.
9. $15\sqrt{6} - \sqrt{54} + \sqrt{24}$.
10. $\frac{1}{3}\sqrt{108} - 7\frac{1}{5}(72)^{\frac{1}{2}}$.
11. $\frac{1}{3}(\frac{1}{54})^{-\frac{1}{3}} + 4(\frac{1}{128})^{-\frac{1}{3}}$.
12. $3\sqrt[3]{\frac{4}{27}} - 2\sqrt[3]{\frac{1}{2}} - \frac{1}{4}\sqrt[3]{256}$.
13. $\frac{1}{2}\sqrt{128} - \sqrt{18} - \frac{1}{3}\sqrt{8}$ -5.
14. $\sqrt{3} + \sqrt{1\frac{1}{3}} - 2\sqrt{5\frac{1}{3}} + 7\sqrt{\frac{1}{3}}$.
15. $\sqrt{9\,mn^3} - \sqrt{m^3n} + \sqrt{m^3n - 6\,m^2n^2 + 9\,mn^3}$.
16. $2\sqrt{125} + \sqrt[3]{81} - (-512^{\frac{1}{3}}) - 4\sqrt[4]{\frac{25}{16}} - \frac{1}{2}\sqrt[3]{192}$.

17. $a\sqrt[6]{4a^2} + b\sqrt[3]{16a}$.

18. $3\sqrt{(\frac{1}{20})^{-1}} + \frac{3}{7}\sqrt{(\frac{1}{45})^{-1}} - \sqrt{(\frac{5}{4})^{-1}}$.

19. $\frac{1}{6}(72)^{\frac{1}{3}} - 3(\frac{1}{3})^{\frac{1}{3}}$.

20. $\frac{2}{3}\sqrt{\frac{2}{5}} + \sqrt{\frac{1}{10}} + \frac{5}{4}\sqrt{\frac{1}{40}}$.

21. $\sqrt{84ab^2} + b\sqrt{75a} + \sqrt{3a(a-9b)^2}$.

22. $\sqrt[m]{a^{2m}b^{mn}c^{m^2+1}} - \sqrt[m]{a^{3m^2}c^{mn+1}} + \sqrt[m]{cx^{mn-m}}$.

23. $\sqrt[s]{25a^{s+t}b^{t-s}} + \sqrt[s]{5^{s+2}a^tb^{st+t}} - \sqrt[s]{\left(\frac{1}{5a^s}\right)^{-2}\left(\frac{1}{b^{-t}}\right)}$.

SECTION III

MULTIPLICATION

395. PROBLEM 6. **To multiply together radical expressions.**

Rule. *Reduce the radicals to the same degree. Then multiply the coefficients together for a new coefficient, and the numbers under the radical signs for a new expression under the common sign.*

Dem. See Problem 4 in reduction and law 2 of Evolution.

QUESTIONS. What is Problem 4 and its demonstration? Prove that the product of the nth roots of several factors equals the nth root of their product. What are similar radicals? Must radicals be similar in order to be multiplied together? State the Commutative Law; the Associative Law; the Distributive Law.

MODEL SOLUTIONS

1. $\sqrt[3]{2} \times \sqrt{3} = \sqrt[6]{2^2} \times \sqrt[6]{3^3} = \sqrt[6]{4 \times 27} = \sqrt[6]{108}$. Why?

2. $\sqrt[3]{\frac{2}{3}} \cdot \sqrt[5]{\frac{1}{2}} = \frac{1}{3}\sqrt[3]{2 \cdot 3^2} \cdot \frac{1}{2}\sqrt[5]{2^4}$

$= \frac{1}{6}\sqrt[15]{2^5 \cdot 3^{10}} \cdot \sqrt[15]{2^{12}} = \frac{1}{6}\sqrt[15]{2^{17} \cdot 3^{10}}$

$= \frac{1}{6}\sqrt[15]{2^{15} \cdot 2^2 \cdot 3^{10}} = \frac{1}{3}\sqrt[15]{2^2 \cdot 3^{10}}$. Why?

3. $(\sqrt{a} - \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - 2\sqrt{ab} + b$.

EXAMPLES

1. $\sqrt{8}\cdot\sqrt{18}$.

2. $\sqrt{6}\cdot\sqrt[3]{\frac{1}{12}}$.

3. $\sqrt[3]{3a}\cdot\sqrt{2b}$.

4. $\sqrt[3]{b^n}\cdot\sqrt{a^m}$.

5. $\sqrt[3]{3}\cdot\sqrt[3]{2}\cdot\sqrt[6]{\frac{1}{3}}$.

6. $\frac{1}{3}\sqrt[3]{x^2}\cdot 6\sqrt{x^3}$.

7. $\sqrt[3]{\frac{2}{3}}\cdot\sqrt[4]{\frac{3}{4}}$.

8. $\sqrt[5]{\frac{3}{8}}\cdot\sqrt[3]{\frac{2}{3}}$.

9. $\sqrt[4]{\frac{2}{3}}\cdot\sqrt[5]{\frac{2}{3}}$.

10. $\sqrt{a+b}\cdot\sqrt{a-b}$.

11. $\sqrt[4]{3a^2cx^3}\cdot\sqrt[3]{6ac^2x^2}$.

12. $\left(\frac{x}{3}\sqrt{\frac{1}{3}}-\frac{3}{x}\sqrt{3}\right)^2$.

13. $(2+3\sqrt{5})(2-3\sqrt{5})$.

14. $(\sqrt{2}-\sqrt{3})(\sqrt{2}-\sqrt{3})$.

15. $\sqrt{1+x^2}\cdot\sqrt[3]{1+x^2}$.

16. $\sqrt{2}\cdot\sqrt[3]{3}\cdot\sqrt[4]{\frac{1}{2}}\cdot\sqrt[3]{\frac{1}{3}}$.

17. $\sqrt[7]{\frac{x^2}{y^3}}\cdot\sqrt[12]{\frac{y^3}{x^2}}\cdot\frac{x}{y}\sqrt[3]{\frac{x}{y}}$.

18. $\sqrt[3]{(a+b)^5}\cdot\sqrt[3]{(a+b)^{-2}}$.

19. $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$.

20. $(2\sqrt{3}-3\sqrt{2})(3\sqrt{3}+2\sqrt{2})$.

21. $(x^n+\sqrt{x^n}+1)(x^{-n}-\sqrt{x^{-n}}+1)$.

22. $\left(\frac{x}{2}\sqrt{2}-\frac{y}{3}\sqrt{3}\right)\left(\frac{x}{2}\sqrt{2}+\frac{y}{3}\sqrt{3}\right)$.

23. $(2+\sqrt{3}-\sqrt[3]{4})(2-\sqrt{3}+\sqrt[3]{4})$.

24. $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{a+\sqrt{a^2-x^2}}\cdot\dfrac{x}{a-\sqrt{a^2-x^2}}$.

25. $(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}+\sqrt{b}+\sqrt{c})$.

26. $(\sqrt{2}-\sqrt{15}+2)(-\sqrt{2}+\sqrt{15}+-5)$.

27. $\sqrt[m]{a+b}\cdot\sqrt[n]{a+b}\cdot\sqrt[m]{a-b}\cdot\sqrt[n]{a-b}$.

28. $(2\sqrt{a}+\sqrt{b-3a})(2\sqrt{a}-\sqrt{b-3a})$.

29. $(\sqrt[3]{16}-3+\sqrt[3]{54})(\sqrt[3]{18}+2-\sqrt[3]{24})$.

30. $(\sqrt{10}+\sqrt{15}-\sqrt{5})(\sqrt{10}-\sqrt{15}-\sqrt{5})$.

Section IV

DIVISION

396. Problem 7. **To divide one radical by another.**

Rule. *Reduce the radicals to the same degree. Then divide the coefficients for a new coefficient, and the numbers under the radical signs for a new expression under the common sign.*

Dem. See Problem 4 in reduction and law 3 of Evolution. Review each.

Prove that

the nth root of a divided by the nth root of b $= \sqrt[n]{\dfrac{a}{b}}$.

Model Solutions

1. $\sqrt{2} \div \sqrt[3]{2} = \sqrt[6]{2^3} \div \sqrt[6]{2^2} = \sqrt[6]{2}$.

2. $\sqrt{2} \div \sqrt[3]{3} = \sqrt[6]{2^3} \div \sqrt[6]{3^2} = \sqrt[6]{\dfrac{2^3}{3^2}} = \sqrt[6]{\dfrac{2^3 \cdot 3^4}{3^6}} = \dfrac{1}{3}\sqrt[6]{2^3 \cdot 3^4}$.

3. $\sqrt[3]{4x^3y^2z} \div \sqrt[5]{2x^2y^5z^3} = \sqrt[15]{2^{10}x^{15}y^{10}z^5} \div \sqrt[15]{2^3x^6y^{15}z^9} = \sqrt[15]{\dfrac{2^7x^9}{y^5z^4}}$

$$= \frac{1}{yz}\sqrt[15]{2^7x^9y^{10}z^{11}}.$$

4. $(x + \sqrt{xy} + y) \div (\sqrt{x} - \sqrt[4]{xy} + \sqrt{y}) = \sqrt{x} + \sqrt[4]{xy} + \sqrt{y}$, by factoring. State the principle by which $x + \sqrt{xy} + y$ is factored; also $x^{4n} + y^{4n}$.

BY LONG DIVISION

$$\begin{array}{l|l}
x + \sqrt{xy} + y & \sqrt{x} - \sqrt[4]{xy} + \sqrt{y} \\
\underline{x - \sqrt[4]{x^3y} + \sqrt{xy}} & \overline{\sqrt{x} + \sqrt[4]{xy} + \sqrt{y}} \\
\qquad \sqrt[4]{x^3y} + y & \\
\qquad \underline{\sqrt[4]{x^3y} - \sqrt{xy} + \sqrt[4]{xy^3}} & \\
\qquad\qquad \sqrt{xy} - \sqrt[4]{xy^3} + y & \\
\qquad\qquad \underline{\sqrt{xy} - \sqrt[4]{xy^3} + y} &
\end{array}$$

5. $(2\sqrt{7}-3\sqrt{5})\div(3\sqrt{7}+2\sqrt{5})=\dfrac{2\sqrt{7}-3\sqrt{5}\cdot(3\sqrt{7}-2\sqrt{5})}{3\sqrt{7}+2\sqrt{5}\cdot(3\sqrt{7}-2\sqrt{5})}$

$$=\frac{72-13\sqrt{35}}{63-20}=\frac{72}{43}-\frac{13}{43}\sqrt{35}.$$

Why was $3\sqrt{7}+2\sqrt{5}$ multiplied by $3\sqrt{7}-2\sqrt{5}$? Why does this operation free the denominator from radicals?

Why was $2\sqrt{7}-3\sqrt{5}$ multiplied by $3\sqrt{7}-2\sqrt{5}$? Why does not this operation free the numerator from radicals?

EXAMPLES

1. $\sqrt{54}\div\sqrt[4]{36}.$
2. $\sqrt[6]{\dfrac{a^9}{27}}\div\sqrt{6a^5}.$
3. $4\sqrt{18}\div 2\sqrt[3]{81}.$
4. $\frac{3}{14}\sqrt[3]{\frac{2}{3}}\div\frac{3}{21}\sqrt[3]{\frac{9}{4}}.$
5. $\sqrt[4]{8a^3b^2}\div 2a^2b^2.$
6. $\sqrt{2\sqrt[3]{2}}\div\sqrt{\sqrt{2\sqrt[3]{2}}}.$
7. $\sqrt{\dfrac{x^2}{c^3}}\div\sqrt[3]{\dfrac{x^2}{c^8}}.$
8. $\sqrt[8]{\frac{112}{405}}\div\sqrt[6]{\frac{56}{185}}.$
9. $\sqrt[5]{x^{2a}y^{5m}z^{12c}}\div\sqrt[5]{x^a z^{7c}}.$
10. $(x-1)\div(\sqrt{x}+1).$
11. $\sqrt[3]{x^{-1}\sqrt{y^3}}\div\sqrt{y\sqrt[3]{x}}.$
12. $\sqrt{ab^2x-b^2cx}\div\sqrt{a-c}.$
13. $(\sqrt[4]{a^3}+\sqrt[4]{b^3})\div(\sqrt[4]{a}+\sqrt[4]{b}).$
14. $(\sqrt{x}-\sqrt{y})\div(\sqrt[4]{x}+\sqrt[4]{y}).$
15. $(x^2+xy+y^2)\div(x+\sqrt{xy}+y).$
16. $(x^{4n}+y^{4n})\div(x^{2n}-x^ny^n\sqrt{2}+y^{2n}).$
17. $(\sqrt[5]{a}\cdot\sqrt[4]{b})^3\div\sqrt[3]{c^2}\div\sqrt[6]{c^2b^5}\div\sqrt[5]{a^7}.$
18. $(16\sqrt[3]{18}+24\sqrt[3]{20}-30\sqrt[3]{5})\div 2\sqrt[3]{30}.$
19. $(x+2\sqrt{xy}+y-c)\div(\sqrt{x}+\sqrt{y}+\sqrt{c}).$
20. $(a+b+c-3\sqrt[3]{abc})\div(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}).$

SECTION V

INVOLUTION

397. PROBLEM 8. **To raise a radical to any power.**

Rule. *Raise the coefficient, and also the number under the radical sign, to the required power, and reduce the resulting radical to its simplest form.*

Dem. See the rule for multiplication of radicals, and the laws of Involution.

MODEL SOLUTIONS

1. $(2\sqrt{3})^5 = 2^5\sqrt{3^5} = 2^5\sqrt{3^4 \cdot 3} = 2^5 \cdot 3^2 \cdot \sqrt{3} = 288\sqrt{3}.$
2. $\left(\frac{2}{3}a\sqrt[3]{\frac{3}{4}ax^2}\right)^{10} = \frac{2^{10}}{3^{10}}a^{10}\sqrt[3]{\frac{3^{10}}{2^{20}}a^{10}x^{20}} = \frac{2^4}{3^7}a^{13}x^6\sqrt[3]{\frac{3}{2^2}ax^2} = \frac{8\,a^{13}x^6}{3^7}\sqrt[3]{6\,ax^2}.$
3. $(\sqrt{a} - \sqrt{b} + \sqrt{c})^2 = a + b + c - 2\sqrt{ab} + 2\sqrt{ac} - 2\sqrt{bc}.$
4. $(\sqrt{x} - \sqrt{y})^4 = x^2 - 4\,x\sqrt{xy} + 6\,xy - 4\,y\sqrt{xy} + y^2.$

EXAMPLES

Perform the operation indicated:

1. $(3\sqrt{2})^3.$
2. $(\frac{2}{3}\sqrt[3]{2})^3.$
3. $(a^2\sqrt{a^3})^m.$
4. $(-\frac{1}{2}\sqrt[3]{2})^5.$
5. $(\sqrt{a - x})^4.$
6. $\left(\sqrt{\frac{a-b}{a+b}}\right)^5.$
7. $(-\frac{3}{4}\sqrt[5]{9\,a^2})^4.$
8. $(\sqrt{3\,a^3\sqrt{x}})^2.$
9. $\left(\sqrt[3]{3\,a\sqrt{\frac{2}{a}}}\right)^6.$
10. $(+3\sqrt{a - x})^3.$
11. $(\sqrt[4]{x} - \sqrt[4]{y})^2.$
12. $(\sqrt{a} - \sqrt{b})^3.$
13. $(\sqrt[3]{x} - \sqrt[3]{y})^3.$
14. $(\sqrt{2} - \sqrt{3})^4.$
15. $(\sqrt{3} + \sqrt{2})^5.$
16. $(2\sqrt{3} - 3\sqrt{5})^4.$
17. $\left(a\sqrt{\frac{x}{y}} - b\sqrt{\frac{y}{x}}\right)^2.$
18. $(\sqrt[3]{x} - \sqrt[3]{y} - \sqrt[3]{z})^3.$
19. $(\sqrt{x} - \sqrt{y} + \sqrt{z})^2.$
20. $(\sqrt{a} - \sqrt{b} + \sqrt{c})^3.$
21. $(\sqrt{a} - \sqrt{b} - \sqrt{c})^4.$

22. $(-2\sqrt[m]{a^{s+m}})^{s-m}$.

23. $\left(2x\sqrt{\dfrac{3(x-y)}{2(x+y)^3}}\right)^3$.

24. $(-3ab^2\sqrt[n]{a^nb^mc^{mn}})^{mn}$.

25. $(\sqrt{a-x}+\sqrt{a+x})^4$.

26. $(\sqrt{a+x}-\sqrt{a-x})^3$.

27. $\left(\sqrt{\dfrac{x}{2}-4}-\sqrt{\dfrac{x}{2}+4}\right)^2$.

SECTION VI

EVOLUTION

398. PROBLEM 9. **To extract any root of a radical number.**

Rule. *Extract the required root of the coefficient, and also of the quantity under the radical sign, and reduce the resulting radical to its simplest form.*

Dem. See the laws of Evolution.

MODEL SOLUTIONS

1. Extract the cube root of $-8a^6\sqrt{-27a^{-30}b^{12}}$.

$$\sqrt[3]{-8a^6\sqrt{-27a^{-30}b^{12}}} = \sqrt[3]{-8a^6}\cdot\sqrt[3]{\sqrt{-27a^{-30}b^{12}}}. \quad \text{Why?}$$
$$= -2a^2\sqrt{-3a^{-10}b^4}. \quad \text{Why?}$$
$$= -2a^{-3}b^2\sqrt{-3}. \quad \text{Why?}$$
$$= -\frac{2b^2}{a^3}\sqrt{-3}. \quad \text{Why?}$$

2. $\sqrt[3]{3\sqrt{3}}=\sqrt[3]{\sqrt{9\cdot 3}}=\sqrt[3]{\sqrt{27}}=\sqrt{\sqrt[3]{27}}=\sqrt{3}$. Why?

EXAMPLES

1. $\sqrt[3]{\frac{1}{27}x^6\sqrt{y}}$.

2. $\sqrt[4]{25b^4\sqrt{y}}$.

3. $\sqrt[5]{224\sqrt[3]{3x^4}}$.

4. $\sqrt[3]{\dfrac{x}{5}\sqrt{\dfrac{5}{x}}}$.

5. $\sqrt{\sqrt{\dfrac{x\sqrt{y}}{\sqrt[3]{xy}}}}$.

6. $\sqrt{27\sqrt[3]{135x^6y^4}}$.

7. $\sqrt[5]{486x^8\sqrt[3]{2^2x^2}}$.

8. $\sqrt{(1-x)\sqrt{1-x}}$.

9. $\sqrt{(4x^2y-12xz\sqrt{xy}+9xz^2)}$. Check.

10. $\sqrt[3]{(x-3\sqrt[3]{x^2y}+3\sqrt[3]{xy^2}-y)}$. Check.

11. $\sqrt{(x^{-2}-2\sqrt[3]{x^{-5}}+\sqrt[3]{x^{-4}}+2x^{-1}-2\sqrt[3]{x^{-2}}+1)}$.

SECTION VII

RATIONALIZATION

399. To **rationalize** an expression is to free it from radical signs and fractional exponents.

400. PROBLEM 10. **To rationalize a monomial denominator.**

Rule. *Represent by a fractional exponent the indicated power and root of the quantity to be rationalized. Then multiply both terms of the fraction by that quantity affected with an exponent, which, added to the given fractional exponent, makes the latter integral.*

Dem. 1. Explain the first step of the rule.

2. Why is the value of the fraction not changed?

3. Why is it evident that the denominator is rationalized by the process set forth in the rule?

MODEL SOLUTIONS

1. $\dfrac{15}{\sqrt[3]{a}}=\dfrac{15\cdot a^{\frac{2}{3}}}{a^{\frac{1}{3}}\cdot a^{\frac{2}{3}}}=\dfrac{15\sqrt[3]{a^2}}{a}.$

2. $\dfrac{a}{\sqrt[r]{b}}=\dfrac{a\cdot b^{\frac{r-1}{r}}}{b^{\frac{1}{r}}\cdot b^{\frac{r-1}{r}}}=\dfrac{a\sqrt[r]{b^{r-1}}}{b}.$

3. $\dfrac{\sqrt{x}}{\sqrt[s]{x^m y^{2a}}}=\dfrac{\sqrt{x}\cdot x^{\frac{s-m}{s}}y^{\frac{s-2a}{s}}}{x^{\frac{m}{s}}y^{\frac{2a}{s}}\cdot x^{\frac{s-m}{s}}y^{\frac{s-2a}{s}}}=\dfrac{\sqrt{x}\cdot\sqrt[s]{x^{s-m}y^{s-2a}}}{xy}=\dfrac{\sqrt[2s]{x^{3s-2m}y^{2s-4a}}}{xy}.$

EXAMPLES

1. $\dfrac{2}{\sqrt{3}}.$

2. $\dfrac{5}{\sqrt{a^3b}}.$

3. $\dfrac{1}{2\sqrt[3]{a^2}}.$

4. $\dfrac{\sqrt[4]{a}}{3\sqrt[4]{2}}.$

5. $\dfrac{3}{\sqrt[5]{a^2x}}.$

6. $\dfrac{7}{2\sqrt[7]{2}}.$

7. $\dfrac{a}{\sqrt[3]{a-x}}.$

8. $\dfrac{1}{\sqrt{3a^3x^5}}.$

9. $\dfrac{5}{\sqrt{7}}.$

10. $\dfrac{z}{\sqrt[n]{z^m}}.$

11. $\dfrac{1}{\sqrt{2}\sqrt{a}}.$

12. $\dfrac{b}{\sqrt[n]{a-b}}.$

13. $\dfrac{1}{\sqrt[s]{x^{2r}y^{3t}}}.$

14. $\dfrac{1}{\sqrt[3]{x^m y^n z^s}}.$

15. $\dfrac{3x}{\sqrt[n]{a^3b^{\frac{2}{3}}c^m}}.$

16. $\dfrac{2a}{3a\sqrt[r]{a b^{2q}}}.$

401. PROBLEM 11. **To rationalize a binomial denominator of the second degree.**

Rule. *Multiply both terms of the fraction by the denominator with one of its signs changed.*

Dem. By this process the denominator is rationalized because each square root is squared, both terms being thus made rational. What special theorem is involved? Does squaring both terms of the fraction rationalize a binomial denominator? Why? Is the value of the fraction changed? Why?

MODEL SOLUTIONS

1. $$\frac{a}{x+\sqrt{y}} = \frac{a(x-\sqrt{y})}{(x+\sqrt{y})(x-\sqrt{y})} = \frac{ax-a\sqrt{y}}{x^2-y}.$$

2. $$\frac{\sqrt{b}}{\sqrt{x}-\sqrt{y}} = \frac{\sqrt{b}(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{\sqrt{bx}+\sqrt{by}}{x-y}.$$

3. $$\frac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}} = \frac{(\sqrt{x+a}-\sqrt{x-a})(\sqrt{x+a}-\sqrt{x-a})}{(\sqrt{x+a}+\sqrt{x-a})(\sqrt{x+a}-\sqrt{x-a})}$$

$$= \frac{x+a-2\sqrt{x^2-a^2}+x-a}{(x+a)-(x-a)} = \frac{2x-2\sqrt{x^2-a^2}}{2a} = \frac{x-\sqrt{x^2-a^2}}{a}.$$

EXAMPLES

1. $\dfrac{2}{\sqrt{3}-\sqrt{2}}$.

2. $\dfrac{3}{2\sqrt{5}-3\sqrt{3}}$.

3. $\dfrac{2-\sqrt{5}}{2+\sqrt{5}}$.

4. $\dfrac{x}{\sqrt{x}+\sqrt{y}}$.

5. $\dfrac{3\sqrt{7}-5\sqrt{5}}{2\sqrt{7}+3\sqrt{5}}$.

6. $\dfrac{a\sqrt{b}+b\sqrt{a}}{a\sqrt{b}-b\sqrt{a}}$.

7. $\dfrac{a-x}{\sqrt{a-x}+\sqrt{a+x}}$.

8. $\dfrac{\sqrt{x^2-1}+\sqrt{x^2+1}}{\sqrt{x^2-1}-\sqrt{x^2+1}}$.

9. $\dfrac{\sqrt{2}}{\sqrt{3}}+\dfrac{\sqrt{5}}{\sqrt{2}}$.

10. $\dfrac{\sqrt{x}+\sqrt{x+y}}{\sqrt{x}-\sqrt{x+y}}$.

11. $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}-\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.

12. $\dfrac{\sqrt{2}+1}{\sqrt{3}-1}+\dfrac{\sqrt{2}-1}{\sqrt{3}+1}$.

13. $\dfrac{2-3\sqrt{\frac{2}{3}}}{\frac{1}{3}\sqrt{\frac{1}{2}}+\frac{2}{3}\sqrt{\frac{3}{5}}}$.

14. $\dfrac{x-\sqrt{x^2+1}}{x+\sqrt{x^2+1}}$.

15. $\dfrac{1+\sqrt{1+x}}{1+\sqrt{1-x}}$.

16. $\dfrac{1+x}{(1+x)+\sqrt{1+x^2}}$.

17. $\dfrac{2}{x+\sqrt{2-x^2}}$.

18. $\dfrac{a^2-x^2}{a-\sqrt{a^2-x^2}}$.

402. PROBLEM 12. **To rationalize a trinomial denominator of the second degree.**

Rule. *Multiply both terms of the fraction by the denominator with the sign of one of its radicals changed. Repeat this process until the denominator becomes rational.*

Dem. The same as for Problem 11.

MODEL SOLUTION

$$\frac{3}{\sqrt{3}+2\sqrt{5}-\sqrt{2}}=\frac{3(\sqrt{3}+2\sqrt{5}+\sqrt{2})}{(\sqrt{3}+2\sqrt{5}-\sqrt{2})(\sqrt{3}+2\sqrt{5}+\sqrt{2})}=\frac{(3\sqrt{3}+2\sqrt{5}+\sqrt{2})}{3+4\sqrt{15}+20-2}$$

$$=\frac{3(\sqrt{3}+2\sqrt{5}+\sqrt{2})}{21+4\sqrt{15}}=\frac{3(\sqrt{3}+2\sqrt{5}+\sqrt{2})(21-4\sqrt{15})}{(21+4\sqrt{15})(21-4\sqrt{15})}$$

$$=\frac{3(\sqrt{3}+2\sqrt{5}+\sqrt{2})(21-4\sqrt{15})}{441-240}=\frac{21\sqrt{2}-19\sqrt{3}+30\sqrt{5}-4\sqrt{30}}{67}.$$

EXAMPLES

1. $\dfrac{1}{\sqrt{1}+\sqrt{2}-\sqrt{3}}$.

2. $\dfrac{2}{\sqrt{2}-\sqrt{3}+\sqrt{5}}$.

3. $\dfrac{3}{2-3\sqrt{3}-5\sqrt{5}}$.

4. $\dfrac{\sqrt{2}+\sqrt{3}-1}{\sqrt{2}-\sqrt{3}+1}$.

5. $\dfrac{2-\sqrt{2}-\sqrt{6}}{2+\sqrt{2}+\sqrt{6}}$.

6. $\dfrac{4}{\sqrt{2}-\sqrt{5}-\sqrt{3}}$.

7. $\dfrac{3}{2\sqrt{2}-3\sqrt{3}-1}$.

8. $\dfrac{\sqrt{5}+\sqrt{6}+\sqrt{8}}{\sqrt{2}-\sqrt{3}+\sqrt{5}}$.

SYNOPSIS FOR REVIEW, CHAPTER XIV

RADICAL EXPRESSIONS

Sect. I — Definitions and Reduction

- Radical number, surd, root, degree of root, index, rational, irrational, imaginary, **386**. Reduction of Radicals, **387**. Similar Radicals, **388**.
- Probs.
  1. To reduce a radical like $\sqrt[n]{a^n b}$ to its simplest form, **389**. Rule. Dem.
     Cor. A surd fraction simplified, **390**.
  2. To introduce a coefficient, **391**. Rule. Dem.
  3. To simplify an expression of the form $\sqrt[mn]{a^m}$, **392**. Rule. Dem.
  4. To reduce radicals to the same degree, **393**. Rule. Dem.

Sect. II — Addition and Subtraction

Prob. 5. To add or subtract radicals, **394**. Rule. Dem.

Sect. III — Multiplication

Prob. 6. To multiply together radical expressions, **395**. Rule. Dem.

Sect. IV — Division

Prob. 7. To divide one radical by another, **396**. Rule. Dem.

Sect. V — Involution

Prob. 8. To raise a radical to any power, **397**. Rule. Dem.

Sect. VI — Evolution

Prob. 9. To extract any root of a radical number, **398**. Rule. Dem.

Sect. VII — Rationalization

- Meaning of the expression "to rationalize," **399**.
- Probs.
  1. To rationalize a monomial denominator, **400**. Rule. Dem.
  2. To rationalize a binomial denominator of the 2d degree, **401**. Rule. Dem.
  3. To rationalize a trinomial denominator of the 2d degree, **402**. Rule. Dem.

Sample Test Questions

1. Define radical number, surd, root, degree of root, index, similar radicals.
2. How is $\sqrt{24}$ simplified? $\sqrt[6]{9}$? $\sqrt[3]{\frac{1}{4}}$? State the laws involved.
3. How is a coefficient introduced? State the reasons for the steps taken.
4. How are radicals reduced to the same degree? State the reasons.
5. How are radicals added? Multiplied? Divided? State the laws involved.
6. How are radicals raised to required powers? How are required roots of radicals extracted?
7. How is a monomial denominator rationalized? A binomial denominator? A trinomial denominator?

CHAPTER XV

IMAGINARY AND COMPLEX NUMBERS

SECTION I

DEFINITIONS AND MODEL OPERATIONS

403. An **Imaginary Number** is an indicated even root of a negative number.

ILLUSTRATIONS. $\sqrt{-1}$, $2\sqrt{-3}$, $a\sqrt{-c}$, $\sqrt[4]{-a}$.

404. A **Pure Imaginary** is of the form $\sqrt{-c}$ and is used only as a factor. Its square is $-c$.

ILLUSTRATION. $\sqrt{-c} \cdot \sqrt{-c} = (-c)^{\frac{1}{2}} \cdot (-c)^{\frac{1}{2}}$

$$= (-c)^{\frac{2}{2}} = -c; \text{ or,}$$

$$\sqrt{-c} \cdot \sqrt{-c} = \sqrt{(-c)^2} = -c; \text{ or,}$$

$$\sqrt{-c} \cdot \sqrt{-c} = \sqrt{c^2} = -c, \text{ but not } +c.$$

For $-c$ is one of the two equal factors which were multiplied together to produce c^2.

The form $\sqrt{-c} = \sqrt{c} \cdot \sqrt{-1}$, by law 2 of Evolution. It is generally written $i\sqrt{c}$ by substituting i for $\sqrt{-1}$.

405. A **Complex Number** is of the form $a + \sqrt{-c}$, and is the indicated sum of a real and a pure imaginary number.

406. Two complex numbers are **conjugate** to each other when their real parts are the same and their imaginary parts differ only in sign. Thus $a + bi$ is conjugate to $a - bi$.

407. The application of the rules for the **Fundamental Principles** is shown by the following:

Models of Operation

1. $(a+bi)+(c+di)=(a+c)+(b+d)i.$
2. $(a+bi)-(c+di)=(a-c)+(b-d)i.$
3. $(a+bi)\cdot(c+di)=(ac-bd)+(ad+bc)i.$
4. $(a+bi)\div(c+di)=\dfrac{a+bi\cdot(c-di)}{c+di\cdot(c-di)}=\dfrac{(ac+bd)+(bc-ad)i}{c^2+d^2}.$
5. $(a+bi)^{-1}=\dfrac{1}{a+bi}=\dfrac{a-bi}{a^2+b^2}.$ Why?
6. $(a+bi)^2=a^2+2bi-b^2.$

Section II

ADDITION AND SUBTRACTION

408. Problem 1. **To add or subtract complex numbers.**

Rule. *Reduce them to the form* $a+b\sqrt{-1}$, *add or subtract the real parts and the imaginary parts separately, and combine the results with the proper sign.*

Dem. See the laws for addition and subtraction of simple numbers.

Model Solution

$$\begin{array}{rr} 3+\sqrt{-4}=3+2\sqrt{-1}= & 3+2i \\ 5-2\sqrt{-9}=5-6\sqrt{-1}= & \underline{8-6i} \\ \text{Sum}= & 11-4i. \end{array}$$

EXAMPLES

1. $(2+\sqrt{-1})+(3-\sqrt{-64})$.
2. $(2+\sqrt{-1})-(3-\sqrt{-64})$.
3. $(a+\sqrt{-b})+(a+\sqrt{-c})$.
4. $(c-\sqrt{-b})-(a-\sqrt{-c})$.
5. $(7-3\sqrt{-4})+(8+5\sqrt{-16})$.
6. $(17+5\sqrt{-8})-(14-3\sqrt{-18})$.
7. $(5+3\sqrt{-27})+(3-5\sqrt{-12})$.
8. $(4\sqrt{-27}-2)-(3+2\sqrt{-16})$.
9. $(\sqrt{-x}+1)-(1-\sqrt{-y})$.
10. $(2\sqrt{-5}+x)-(1-\sqrt{-y})$.

SECTION III

MULTIPLICATION

409. PROBLEM 2. **To multiply together complex numbers.**

Rule. *Reduce them to the form* $a+b\sqrt{-1}$, *and multiply as in common radicals, observing that* $\sqrt{-1} \times \sqrt{-1} = -1$.

Dem. Let the student state the reasons for each step.

MODEL SOLUTIONS

1.
$$\begin{array}{l}
3 - 2\sqrt{-4} = 3 - 4\sqrt{-1} = 3 - 4i \\
5 + 3\sqrt{-9} = 5 + 9\sqrt{-1} = \underline{5 + 9i} \\
\qquad\qquad\qquad\qquad\qquad\quad 15 - 20i \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{+ 27i - 36i^2} \\
\qquad\qquad\qquad\qquad\qquad\quad 15 + \ \ 7i + 36 = 51 + 7i.
\end{array}$$

2.
$$\begin{aligned}
4\sqrt{-3} \cdot -2\sqrt{-2} \cdot \sqrt{-4} &= (4 \cdot -2 \cdot 2)(\sqrt{3} \cdot \sqrt{2} \cdot \sqrt{1})(\sqrt{-1})^3 \\
&= -16\sqrt{6} \cdot -1 \cdot \sqrt{-1} \\
&= 16\sqrt{6} \cdot \sqrt{-1} \\
&= 16\, i\sqrt{6}.
\end{aligned}$$

EXAMPLES

1. $(a+bi)^2$.
2. $(a-bi)^2$.
3. $(\frac{1}{2} - \frac{1}{2}\sqrt{-3})^2$.
4. $\sqrt{-a^2} \cdot \sqrt{-b^2}$.
5. $(a+bi)(a-bi)$.
6. $(a+bi)(c+di)$.
7. $(x-yi)(s-ti)$.
8. $-2\sqrt{-2} \cdot -3\sqrt{-3}$.
9. $\sqrt{-256} \cdot \sqrt{-27} \cdot \sqrt{-1}$.
10. $(4i + i\sqrt{2})(2i - i\sqrt{3})$.
11. $\sqrt{-2} \cdot \sqrt{-8} \cdot \sqrt{-18}$.
12. $3\sqrt{-5} \cdot 4\sqrt{-3} \cdot \sqrt{-2}$.
13. $\sqrt{-2} \cdot 4\sqrt{-3} \cdot -2\sqrt{-1}$.
14. $(3 - 2\sqrt{-4})(5 + 3\sqrt{-4})$.
15. $(1 + \sqrt{-1})(1 - \sqrt{-1})$.
16. $(\sqrt{2} + \sqrt{-2})(\sqrt{2} - \sqrt{-2})$.
17. $(3 + 2\sqrt{-3})(3 - 2\sqrt{-3})$.
18. $(5 - 3\sqrt{-1})(5 + 3\sqrt{-1})$.

19. $(\sqrt{-2}-\sqrt{-3}-\sqrt{-4})^2$.

20. $(\sqrt{-a}+\sqrt{-b}+\sqrt{-c})^2$.

21. $(x+y-2\sqrt{-z})(x-y+2\sqrt{-z})$.

22. $(2+\sqrt{-2}+\sqrt{-3})(2+\sqrt{-2}-\sqrt{3})$.

23. $(\sqrt{2}+\sqrt{-2}+\sqrt{-3})(\sqrt{-2}+\sqrt{2}+\sqrt{3})$.

24. $(\sqrt{-2}-\sqrt{-3}+\sqrt{-4})(\sqrt{-4}+\sqrt{-3}-\sqrt{-2})$.

SECTION IV

DIVISION

410. PROBLEM 3. **To divide one complex number by another.**

Rule 1. *Reduce the numbers to the form $a+b\sqrt{-1}$, and divide as in common radicals, observing the principles used in multiplication.*

Dem. Let the student state the laws involved.

Rule 2. *Write the example in the form of a fraction, rationalize the denominator, and simplify.*

Dem. Let the student state the laws involved.

MODEL SOLUTIONS

1. $$\frac{2\sqrt{8}-\sqrt{-10}}{-\sqrt{-2}}=\frac{4\sqrt{2}-\sqrt{10}\cdot\sqrt{-1}}{-\sqrt{2}\cdot\sqrt{-1}}\cdot\frac{\sqrt{2}\cdot\sqrt{-1}}{\sqrt{2}\cdot\sqrt{-1}}$$
$$=\frac{8\sqrt{-1}+\sqrt{20}}{2}=4i+\sqrt{5}.$$

2. $$\frac{5-\sqrt{-2}}{1+\sqrt{-2}}=\frac{5-i\sqrt{2}}{1+i\sqrt{2}}\cdot\frac{(1-i\sqrt{2})}{(1-i\sqrt{2})}$$
$$=\frac{5-6i\sqrt{2}-2}{1+2}=\frac{3-6i\sqrt{2}}{3}=1-2i\sqrt{2}.$$

EXAMPLES

1. $2 \div \sqrt{-2}$.
2. $\sqrt{-a} \div \sqrt{-b}$.
3. $\sqrt{-8} \div -\sqrt{-2}$.
4. $-5\sqrt{-6} \div 2\sqrt{3}$.
5. $2 \div (3-2\sqrt{-2})$.
6. $\sqrt[4]{xy} \div \sqrt{-1}$.
7. $(a+bi) \div (a-bi)$.
8. $(1+\sqrt{-1}) \div (1-\sqrt{-1})$.
9. $(m-\sqrt{-n}) \div (m+\sqrt{-n})$.
10. $(3-5\sqrt{-3}) \div (5+3\sqrt{-3})$.
11. $(\sqrt{-2}-3\sqrt{-6}) \div (\sqrt{-2}+2\sqrt{-5})$.
12. $(2\sqrt{3}-3\sqrt{-2}) \div (3\sqrt{3}+2\sqrt{-2})$.
13. $(3\sqrt{-5}-15\sqrt{-10}-24\sqrt{20}) \div -3\sqrt{-5}$.
14. $(14-\sqrt{15}-7\sqrt{-3}-2\sqrt{-5}) \div (7-\sqrt{-5})$.

SECTION V

INVOLUTION

411. PROBLEM 4. **To determine the general laws for the powers of $\sqrt{-1}$.**

SOLUTION

Let n represent any integer, including 0. Then $4n$, $4n+1$, $4n+2$, and $4n+3$ represent all integral exponents. Hence all forms to be considered are $(\sqrt{-1})^{4n}$, $(\sqrt{-1})^{4n+1}$, $(\sqrt{-1})^{4n+2}$, and $(\sqrt{-1})^{4n+3}$.

1. $(\sqrt{-1})^0 = \quad . \quad . \quad . \quad +1.$
2. $(\sqrt{-1})^1 = \quad \sqrt{-1} = +i.$
3. $(\sqrt{-1})^2 = \quad . \quad . \quad . \quad -1.$
4. $(\sqrt{-1})^3 = -\sqrt{-1} = -i.$
5. $(\sqrt{-1})^4 = \quad (-1)^2 = +1.$

It is noticed that the foregoing table is a cycle of 4 terms. Hence the following laws may be deduced:

Law 1. $(\sqrt{-1})^{4n} = \{(\sqrt{-1})^4\}^n = +1.$
Law 2. $(\sqrt{-1})^{4n+1} = +1 \cdot \sqrt{-1} = +i.$
Law 3. $(\sqrt{-1})^{4n+2} = +1 \cdot -1 = -1.$
Law 4. $(\sqrt{-1})^{4n+3} = -1 \cdot \sqrt{-1} = -i.$

Model Solutions

1. $(2\sqrt{-2})^6 = 2^6 \cdot \sqrt{2^6} \cdot (\sqrt{-1})^{4n+2} = 2^6 \cdot 2^3 \cdot -1 = -2^9.$
2. $(\sqrt{-1})^{999} = i^{4\times 249+3} = i^{4n+3} = -i. \quad n = ?$
3. $(a+b\sqrt{-1})^3 = a^3 + 3a^2bi + 3ab^2i^2 + b^3i^3$
$= a^3 + 3a^2bi - 3ab^2 - b^3i.$

EXAMPLES

1. $(2\sqrt{-3})^5.$
2. $(3\sqrt{-2})^4.$
3. $(\sqrt{-1})^{777}.$
4. $(\sqrt{-8})^{16}.$
5. $(\sqrt{-x^2})^6.$
6. $(\sqrt{-y^2})^{97}.$
7. $i^{4n+500}.$
8. $(a+bi)^2.$
9. $(a-bi)^2.$
10. $(a-bi)^3.$
11. $(a+bi)^4.$
12. $(a-bi)^4.$
13. $(\sqrt{-z^2})^{100}.$
14. $(\sqrt{-w^2})^{1111}.$
15. $(\sqrt{-s^2})^{1001}.$
16. $(\sqrt{-1})^{16+25}.$
17. $(-1 \pm \sqrt{-10})^3.$
18. $(\sqrt{-x}+\sqrt{-y})^4.$
19. $(\sqrt{-2}+\sqrt{-3})^3.$
20. $(\sqrt{-x}-\sqrt{-y}+\sqrt{-2})^2.$

Section VI

EVOLUTION OF QUADRATIC SURDS

412. Prop. If a binomial surd of the form $x+\sqrt{y}$ equals another of the form $a+\sqrt{c}$, then $x=a$, and $\sqrt{y}=\sqrt{c}$.

Dem. 1. $x+\sqrt{y} = a+\sqrt{c}.$

2. $x-a+\sqrt{y} = \sqrt{c}.$

3. $(x-a)^2 + 2(x-a)\sqrt{y} + y = c.$

4. $2(x-a)\sqrt{y} = c - y - (x-a)^2.$

But (4) is not possible unless the coefficient of $\sqrt{y}$ is 0, for a surd cannot equal a rational number.

5. Hence $\quad 2(x-a)=0.$
6. $\quad x-a=0.$
7. $\quad x=a.$

But if $x=a$, (2) becomes $\sqrt{y}=\sqrt{c}$, or,

8. $\quad y=c.$

413. PROBLEM 5. **To extract the square root of a binomial surd of the form $a+\sqrt{\pm c}$.**

SOLUTION

1. Let $\quad \sqrt{a+\sqrt{\pm c}}=\sqrt{x}+\sqrt{y}.$
2. $\quad a+\sqrt{\pm c}=x+y+2\sqrt{xy}.$
3. Hence $\quad x+y=a$, and
4. $\quad 2\sqrt{xy}=\sqrt{\pm c}$, or,
5. $\quad 4xy=\pm c.$

Solving equations (3) and (5) by inspection, and substituting in the second member of (1), the required root is obtained.

Let the student make a rule and a demonstration.

MODEL SOLUTIONS

1. $\sqrt{31+10\sqrt{6}}=\sqrt{x}+\sqrt{y}.$

1. $\quad 31+10\sqrt{6}=x+y+2\sqrt{xy}.$
2. $\quad x+y=31.$
3. $\quad 2\sqrt{xy}=10\sqrt{6}.$
4. $\quad \sqrt{xy}=5\sqrt{6}.$
5. $\quad xy=25\cdot 6.$

Since the sum of two numbers in $(2)=31$ and their product in $(5)=25\cdot 6$, the two numbers are seen to be 25 and 6.

Hence $\sqrt{x}+\sqrt{y}=\sqrt{25}+\sqrt{6}=5+\sqrt{6}.$

The relation of the root to the model may be seen in the following:

$$\begin{array}{ll}
\text{(1) Root} & \text{(2) Model} \\
5+\sqrt{6} & \sqrt{x}+\sqrt{y} \\
\underline{5+\sqrt{6}} & \underline{\sqrt{x}+\sqrt{y}} \\
25+5\sqrt{6} & x+\sqrt{xy} \\
\underline{\quad +5\sqrt{6}+6} & \underline{\quad +\sqrt{xy}+y}
\end{array}$$

$$25+10\sqrt{6}+6=31+10\sqrt{6}\equiv x+2\sqrt{xy}+y$$

2. $\sqrt{2-4\sqrt{-42}}=\sqrt{x}-\sqrt{y}.$

$$\begin{array}{ll}
1. & 2-4\sqrt{-42}=x+y-2\sqrt{xy}. \\
2. & x+y=2. \\
3. & -2\sqrt{xy}=-4\sqrt{-42}. \\
4. & \sqrt{xy}=2\sqrt{-42}. \\
5. & xy=4\cdot -42. \\
6. & xy=2\cdot 2\cdot -6\cdot 7. \\
7. & xy=14\cdot -12. \\
8. & x=14,\ y=-12. \\
9. & \sqrt{x}-\sqrt{y}=\sqrt{14}-\sqrt{-12} \\
10. & \qquad =\sqrt{14}-2\sqrt{-3}.
\end{array}$$

Proof

$$(\sqrt{14}-2\sqrt{-3})^2=14-4\sqrt{-42}-12=2-4\sqrt{-42}.$$

EXAMPLES

1. $\sqrt{4+2\sqrt{3}}.$
2. $\sqrt{4-2\sqrt{3}}.$
3. $\sqrt{6+2\sqrt{5}}.$
4. $\sqrt{12+2\sqrt{11}}.$
5. $\sqrt{22+10\sqrt{-3}}.$
6. $\sqrt{49+12\sqrt{5}}.$
7. $\sqrt{57+12\sqrt{15}}.$
8. $\sqrt{\sqrt{18}-4\frac{1}{2}}.$
9. $\sqrt{7+6\sqrt{-2}}.$
10. $\sqrt{-2-2\sqrt{-15}}.$
11. $\sqrt{7+4\sqrt{3}}.$
12. $\sqrt{14-4\sqrt{6}}.$
13. $\sqrt{1+4\sqrt{-3}}.$
14. $\sqrt{41-12\sqrt{5}}.$
15. $\sqrt{67-12\sqrt{30}}.$

Section VII

EXERCISES IN INDICES AND RADICALS

MISCELLANEOUS EXAMPLES

1. Remove the literal factors to the numerator:

$$\frac{a^3}{3\,b^2c^2}+\frac{4\,c^2}{a^2b}+\frac{2}{ab^{-1}c^{-1}}+\frac{1}{3\,abc}+\frac{6}{a^{-\frac{5}{4}}b^{-\frac{1}{6}}c^{\frac{7}{8}}}.$$

2. Remove the literal factors to the denominator:

$$a^{-2}b^{-2};\ \ \frac{a^2}{b};\ \ \frac{7\,ab}{3\,x};\ \ \frac{9\,x^{-4}y^{-2}}{a^2};\ \ 6\,a^{-\frac{5}{4}}b^{-\frac{1}{6}}c^{-\frac{7}{8}}.$$

3. Express with radicals:

$$a^{\frac{2}{3}};\ 5\,a^3b^{\frac{1}{2}};\ 7\,x^{\frac{1}{2}}y^{\frac{3}{2}};\ 2\,y^{\frac{3}{4}};\ x^{\frac{2}{4}}y^{\frac{2}{4}}z^{\frac{1}{4}};\ a^{\frac{3}{4}}b^{\frac{6}{7}}c^{\frac{12}{5}}.$$

4. Express with fractional exponents:

$$\sqrt{a^3};\ \sqrt[3]{x^2y^5};\ \sqrt[8]{x^4y^6c};\ \sqrt[5]{a^2b^{-3}\sqrt{c^{-1}}}.$$

5. Express with positive indices:

$$a^{-2}b^{-3}c-ax^{-2}+x^{-4}y^{-5}+3\,a^{-7}b^{-\frac{1}{3}};\ a^{-2}b^{-1}+\frac{c^{-3}}{x^{-1}b^{-4}}+\frac{c^{-\frac{10}{3}}x^{-5}}{y^{-2}}.$$

Perform the operations indicated:

6. $\left(\dfrac{ay}{x}\right)^{\frac{1}{2}}\cdot\left(\dfrac{bx}{y^2}\right)^{\frac{1}{3}}\cdot\left(\dfrac{y^3}{a^3b^2}\right)^{\frac{1}{6}}\cdot\left(\dfrac{x^3y^{-3}}{x^{-3}y^3}\right)^{\frac{4}{3}}.$

7. $\left\{\left(\dfrac{1+x}{1-x}\right)^{\frac{3}{2}}-\left(\dfrac{1-x}{1+x}\right)^{\frac{3}{2}}\right\}\div\left(\dfrac{1+x}{1-x}\right)^{\frac{3}{2}}.$

8. $\left(xy^{-2}\cdot\sqrt{xy^3}\cdot\sqrt[3]{xy^4}\cdot\sqrt[4]{xy^5}\cdot\sqrt[5]{x^{-10}y^{-10}}\right)^{\frac{1}{5}}.$

9. $\sqrt[2n]{(a+b)^{m+1}}\cdot\sqrt{(a+b)^{\frac{n+1}{m}}}\cdot\sqrt{(a+b)^{\frac{n-1}{m}}}\cdot\sqrt[m]{(a+b)^{\frac{n-1}{2}}}.$

10. $(2\sqrt{-1})^{123}$.

11. $\dfrac{c\sqrt{ab}-ac}{bc-c\sqrt{ab}}$.

12. $\left(\dfrac{x+a}{\sqrt[3]{x}+\sqrt[3]{a}}\right)^2$.

13. $\dfrac{\sqrt{\sqrt{\frac{1}{2}}-2\sqrt[3]{3}}}{\sqrt{4\sqrt[3]{2}\cdot\sqrt{3}}}$.

14. $\dfrac{\sqrt[4]{a}+\sqrt[4]{b}}{\sqrt[4]{a}-\sqrt[4]{b}}$.

15. $\dfrac{\sqrt[4]{a^3}-\sqrt[4]{b^3}}{\sqrt[4]{a}-\sqrt[4]{b}}$.

16. $\left[\left(\dfrac{a^{-m}}{b^{-n}}\right)^{\frac{p}{m}}\right]^{-\frac{q}{n}}$.

17. $(\sqrt[3]{a}+b^{-2})^{\frac{3}{2}}$.

18. $\left(\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}\right)^3$.

19. $(x^2-2y^{-3})^5$.

20. $\left\{\dfrac{\frac{1}{2}(2)^{\frac{1}{2}}\sqrt[3]{3}}{2\sqrt[4]{2}(3)^{\frac{1}{2}}}\right\}^5$.

21. $\left(\dfrac{1+\sqrt{-1}}{\sqrt{-1}}\right)^3$.

22. $(x+y)\div(x^{\frac{1}{5}}+\sqrt[5]{y})$.

23. $\sqrt[4]{\dfrac{324}{2304}}-\sqrt[6]{\dfrac{8}{421875}}$.

24. $\dfrac{(a+bi)^3-(a-bi)^3}{(a+bi)^2-(a-bi)^2}$.

25. $5\sqrt{2}-1+\dfrac{3\sqrt{2}+2}{5-3\sqrt{2}}$.

26. $\dfrac{\sqrt{14}+\sqrt{12}}{\sqrt{7}-\sqrt{6}}-\dfrac{\sqrt{6}-\sqrt{4}}{\sqrt{3}+\sqrt{2}}$.

27. $a^{\frac{3}{5}}\sqrt{b}\div a^{-\frac{7}{5}}b^{-\frac{1}{4}}c$.

28. $\dfrac{\sqrt{x}}{y^{-\frac{1}{3}}}\left(\dfrac{\sqrt[4]{y}}{x^{\frac{1}{6}}}\right)\div\dfrac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

29. $\sqrt{\left\{\dfrac{(\frac{1}{2})^3+\sqrt{\frac{7}{2}}}{2\sqrt{2}\cdot(\frac{3}{4})^{\frac{1}{2}}}\right\}^4}$.

30. 6th term of $\left(\dfrac{x^{\frac{3}{2}}}{\sqrt{a}}-\dfrac{2\sqrt[6]{y^5}}{b^{\frac{3}{2}}}\right)^8$.

31. 6th term of $\left(\dfrac{6x^2}{7y\sqrt{y}}-\dfrac{y}{\sqrt{3x}}\right)^7$.

32. $$\dfrac{\dfrac{x^2+\sqrt{x^4-a^4}}{x^2-\sqrt{x^4-a^4}}-\dfrac{x^2-\sqrt{x^4-a^4}}{x^2+\sqrt{x^4-a^4}}}{\sqrt[4]{\dfrac{x^2-a^2}{x^2+a^2}}}.$$

33. $$\left(\sqrt{\frac{a^2+\sqrt{a^2-b^2}}{2}}-\sqrt{\frac{a^2-\sqrt{a^2-b^2}}{2}}\right)^2.$$

34. $$\frac{1}{4(1+\sqrt{x})}+\frac{1}{4(1-\sqrt{x})}+\frac{1}{2(1+x)}.$$

35. $$\sqrt{\frac{a^2x-2ax^2+x^3}{a^2+2ax+x^2}}+\sqrt{\frac{a^2x+2ax^2+x^3}{a^2-2ax+x^2}}.$$

36. $2\{x+2+\sqrt{x^2-4}\}\div\{x+2-\sqrt{x^2-4}\}$.

37. $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

38. $\left(\dfrac{+1 \pm \sqrt{-3}}{2}\right)^3$.

39. $x\sqrt[3]{\dfrac{8\,a^4}{27\,b^3}+\dfrac{16\,a^3}{27\,b^2}}$.

40. $\dfrac{a^{-3}}{b^{-\frac{2}{3}}} \div \left(\dfrac{a^{-\frac{1}{4}}b^{\frac{1}{2}}}{a^2c^{-1}}\right)^{-2}$.

41. $\left(\dfrac{a^2+ab+b^2}{a+\sqrt{ab}+b}\right)^2$.

42. $\left(\dfrac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}\right)^2$.

43. $\dfrac{3^{n+2}}{3^{(n+1)n}} \div \dfrac{9^{n+2}}{(3^n)^{n+2}}$.

44. $\dfrac{x^{\frac{4}{3}}-8\,x^{\frac{1}{3}}y}{x^{\frac{2}{3}}+2\sqrt[3]{xy}+4\,y^{\frac{2}{3}}}$.

45. $\left(\dfrac{a^0b^{-1}}{a^{-2}b^3}\right)^5 \div \left(\dfrac{a^{-2}b}{ab^{-2}}\right)^3 \cdot \dfrac{a}{b}\sqrt[3]{\dfrac{a^{-3}b^0}{ab^{-3}}}$.

46. $\sqrt{(1\frac{1}{8})^{-1}}+\sqrt[4]{(20\frac{1}{4})^{-3}}+(30\frac{3}{8})^{\frac{2}{5}}$.

47. $\dfrac{5\,ac}{b^2}\sqrt[3]{ab^2} : \sqrt[4]{\dfrac{9\,c^2}{a^2}} : : x : \dfrac{3\,a^2}{2}\sqrt{\dfrac{3\,c}{ab}}$.

48. $\dfrac{b^2}{a^8} \cdot \dfrac{a^{-1}b^{-2}}{ab^{-3}} \cdot \dfrac{a^2b^{-1}}{b^4} \div \left(\dfrac{a^{-2}-b^{-1}}{a^2}\right)^2$.

49. $\sqrt{31+12\sqrt{-5}}-\sqrt{-1+4\sqrt{-5}}$.

50. $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^3+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^3$.

51. $(x^{-2}-y^{-2}) \div (x^{-\frac{1}{3}}-y^{-\frac{1}{3}})$.

52. $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

53. $(\sqrt[3]{x^2}-2\sqrt[3]{x}+3-2\sqrt[3]{x^{-1}}+\sqrt[3]{x^{-2}})^2$.

54. $\dfrac{(a^{-1}b^{-1}+c^{-1}d^{-1})^2-(a^{-1}c^{-1}+b^{-1}d^{-1})^2}{(c^{-2}-d^{-2})(b^{-2}-c^{-2})}$.

55. $\sqrt{(ax^{-1}-2\,a^{\frac{1}{2}}x^{-\frac{1}{2}}+3-2\,a^{-\frac{1}{2}}x^{\frac{1}{2}}+a^{-1}x)}$.

56. $2\sqrt{75}+\frac{1}{2}\sqrt{147}+\sqrt{\frac{4}{75}}-\sqrt[4]{\frac{9}{16}}-144^{-\frac{1}{4}}$.

57. $\sqrt{bc+2\,b\sqrt{bc-b^2}}+\sqrt{bc-2\,b\sqrt{bc-b^2}}$.

58. If $a=-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, prove $a^2+a+1=0$.

SYNOPSIS FOR REVIEW, CHAPTER XV

IMAGINARY AND COMPLEX NUMBERS	SECT. I **Definitions and Model Operations**	An imaginary number, **403**. A pure imaginary, **404**. A complex number, **405**. Conjugates, **406**. Models of Operation, **407**.
	SECT. II **Addition and Subtraction**	*Prob.* 1. To add or subtract complex numbers, **408**. Rule. Dem.
	SECT. III **Multiplication**	*Prob.* 2. To multiply together complex numbers, **409**. Rule. Dem.
	SECT. IV **Division**	*Prob.* 3. To divide one complex number by another, **410**. Rule. Dem.
	SECT. V **Involution**	*Prob.* 4. To determine the general laws for the powers of $\sqrt{-1}$, **411**.
	SECT. VI **Evolution of Quadratic Surds**	*Prop.* If $x+\sqrt{y}=a+\sqrt{c}$, then $x=a$ and $\sqrt{y}=\sqrt{c}$, **412**. Dem. *Prob.* 5. To extract the square root of a binomial surd, **413**. Rule. Dem.

Sample Test Questions

1. Define an imaginary number; conjugates.
2. What is the form of a pure imaginary? Of a complex number?
3. State the model operations for addition, subtraction, multiplication, division.
4. What is the expansion of $(a+bi)^2$? Of $(a+bi)^{-1}$?
5. How are two complex numbers added or subtracted?
6. How are two complex numbers multiplied or divided?
7. Why is the second rule for division generally better than the first?
8. State the laws for the powers of $\sqrt{-1}$.
9. Show how the laws for the powers of $\sqrt{-1}$ are derived.
10. How is any power of $\sqrt{-1}$ determined? $(\sqrt{-1})^{1223} = ?$
11. If $x+\sqrt{y}=a+\sqrt{c}$, what terms are equal? Prove your statement.
12. How is square root of surd of form $a+\sqrt{\pm c}$ extracted?

CHAPTER XVI

EQUATIONS INVOLVING RADICALS

414. Prop. 1. Raising both members of an equation to the same integral power is a legitimate but not a reversible process in the solution of an equation.

Dem. 1. Let $x = a$.

2. Then $x^2 = a^2$, for any root of (1) will satisfy (2).

But $\sqrt{x^2} = \pm\sqrt{a^2}$, or

$x = \pm a$ is not legitimate, for $x = -a$ is not satisfied for any root of (1).

415. Prop. 2. From every conditional equation a rational and integral equation can be derived; but it does not follow that any root of the new equation will satisfy the original equation, although it generally does so.

ILLUSTRATION 1. $\sqrt{x-2} + \sqrt{x+2} = \sqrt{2}$.

SOLUTION

1. $x - 2 + 2\sqrt{x^2-4} + x + 2 = 2$, by squaring.
2. $2\sqrt{x^2-4} = 2 - 2x$.
3. $\sqrt{x^2-4} = 1 - x$.
4. $x^2 - 4 = 1 - 2x + x^2$.
5. $2x = 5$.
6. $x = \frac{5}{2}$.

Verification

1. $\sqrt{\frac{5}{2}-2}+\sqrt{\frac{5}{2}+2}=\sqrt{2}.$
2. $\sqrt{\frac{1}{2}}+\sqrt{\frac{9}{2}}=\sqrt{2}.$
3. $\frac{1}{2}\sqrt{2}+\frac{3}{2}\sqrt{2}=\sqrt{2}.$
4. $2\sqrt{2}\not\equiv\sqrt{2}.$

Verification by substituting in (2)

1. $\frac{5}{2}-2+2\sqrt{\frac{25}{4}-4}+\frac{5}{2}+2=2.$
2. $5+2\sqrt{\frac{9}{4}}=2.$
3. $5+\sqrt{9}=2.$
4. $5\pm 3=2.$
5. $8\not\equiv 2.$
6. But $2\equiv 2.$

ILLUSTRATION 2. $\sqrt{x+2}-\sqrt{x-2}=\sqrt{2}.$

1. $x+2-2\sqrt{x^2-4}+x-2=2.$
2. $-\sqrt{x^2-4}=1-x.$
3. $x^2-4=x^2-2x+1.$
4. $2x=5.$
5. $x=\frac{5}{2}.$

Verification

1. $\sqrt{\frac{5}{2}+2}-\sqrt{\frac{5}{2}-2}=\sqrt{2}.$
2. $\sqrt{\frac{9}{2}}-\sqrt{\frac{1}{2}}=\sqrt{2}.$
3. $3\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{2}}=\sqrt{2}.$
4. $\sqrt{2}\equiv\sqrt{2}.$

Test the following:

1. $\sqrt{x-2}-\sqrt{x+2}=\sqrt{2}.$
2. $\sqrt{x-1}+\sqrt{x+1}=1.$
3. $\sqrt{x+1}-\sqrt{x-1}=1.$
4. $\sqrt{x-1}-\sqrt{x+1}=1.$

416. PROBLEM. **To rationalize a radical equation.**

Rule. *Arrange the equation so that the largest radical expression shall be one member, and raise both members to the power of the same degree as the radical. If the resulting equation, after simplifying, is not rational, repeat the process.*

Note. Reducing, clearing of fractions, rationalizing the denominators, and applying the principles of Proportion should precede the application of the above rule whenever possible.

MODEL SOLUTIONS

1. Solve $\sqrt{x+9}-\sqrt{x-5}=\dfrac{7}{\sqrt{x+5}}$.

1. $\sqrt{x^2+14x+45}-\sqrt{x^2-25}=7.$
2. $\sqrt{x^2+14x+45}=\sqrt{x^2-25}+7.$
3. $x^2+14x+45=x^2-25+14\sqrt{x^2-25}+49.$
4. $14x+21=14\sqrt{x^2-25}.$
5. $2x+3=2\sqrt{x^2-25}.$
6. $4x^2+12x+9=4x^2-100.$
7. $12x=-109.$
8. $x=-9\frac{1}{12}.$

2. Solve $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\sqrt{m}$ by proportion.

1. $\dfrac{2\sqrt{a+x}}{2\sqrt{a-x}}=\dfrac{\sqrt{m}+1}{\sqrt{m}-1}$, by Art. **301**.
2. $\dfrac{a+x}{a-x}=\dfrac{m+2\sqrt{m}+1}{m-2\sqrt{m}+1}$, by squaring.
3. $\dfrac{2x}{2a}=\dfrac{4\sqrt{m}}{2(m+1)}$, by Art. **301**.
4. $x=\dfrac{2a\sqrt{m}}{m+1}$. Why?

Dem. See Prop. 1.

EXAMPLES

Solve and verify:

1. $\dfrac{3y+\sqrt{4y-y^2}}{3y-\sqrt{4y-y^2}}=\dfrac{2}{1}$.

2. $\dfrac{3\sqrt{x}-4}{2+\sqrt{x}}=\dfrac{15+\sqrt{9x}}{40+\sqrt{x}}$.

3. $\dfrac{\sqrt[3]{x+1}-\sqrt[3]{x-1}}{\sqrt[3]{x+1}+\sqrt[3]{x-1}}=\dfrac{1}{3}$.

4. $\sqrt[m]{a+x}=\sqrt[2m]{x^2+5ax+b^2}$.

5. $\sqrt{x}-\dfrac{35}{\sqrt{x}}-\sqrt{x-21}=0$.

6. $\dfrac{x}{\sqrt{a^2+x^2}}=\dfrac{g-x}{\sqrt{h^2+(g-x)^2}}$.

7. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=m$.

8. $\dfrac{\sqrt[4]{x+1}-\sqrt[4]{x-1}}{\sqrt[4]{x+1}+\sqrt[4]{x-1}}=\dfrac{c}{d}$.

9. $\dfrac{6\sqrt{x}-7}{\sqrt{x}-1}-\dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}=5$.

10. $\sqrt{x-2}-\sqrt{x+2}=\sqrt{-22}$.

11. $\dfrac{2s+1}{4\sqrt{3s}-1}-\dfrac{\sqrt{3s}-4}{6}=1$.

12. $\dfrac{5Q-1}{\sqrt{5Q}+1}-\dfrac{\sqrt{5Q}-1}{2}=1$.

13. $\sqrt{x+5}+\sqrt{x-8}=\sqrt{-3}$.

14. $\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}=\sqrt{-1}$.

15. $\dfrac{z-2}{\sqrt{z}+\sqrt{2}}=\dfrac{\sqrt{z}-\sqrt{2}}{3}+\sqrt{8}$.

16. $x^{\frac{1}{2}}+(x-9)^{\frac{1}{2}}=36(x-9)^{-\frac{1}{2}}$.

17. $\sqrt{\dfrac{x}{a}+\dfrac{c}{b}}+\sqrt{\dfrac{x}{a}-\dfrac{c}{b}}=\sqrt{\dfrac{4x}{a}-\dfrac{2c}{b}}$.

18. $1-\sqrt{1-w}=E(1+\sqrt{1-w})$.

19. $\sqrt{3t+7}-\dfrac{4}{\sqrt{3t+7}}-\sqrt{3t}=0$.

20. $\dfrac{5}{s+\sqrt{s^2+5}}-\dfrac{5}{\sqrt{5+s^2}-s}=s+8$.

21. $\dfrac{243+324\sqrt{3x}}{16x-3}=16x-8\sqrt{3x}+3$.

CHAPTER XVII

QUADRATIC EQUATIONS

SECTION I

METHODS OF SOLVING

417. A **Quadratic Equation** is an equation of the second degree.

418. A **Complete Polynomial** is one of the form

$$Ax^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + \cdots + L.$$

419. An **Incomplete Polynomial** is one in which one or more of the terms after Ax^n are missing, as $x^3 + 2x$, $x^4 + 1$, or $x^2 - 1$.

420. The **General Type** of a quadratic equation is

$$Ax^2 + Bx + C = 0.$$

421. A **Complete** quadratic equation is of the form

$$Ax^2 + Bx + C = 0.$$

422. An **Incomplete** quadratic equation is one in which one or more of the terms after Ax^2 are missing, as $Ax^2 = 0$, $Ax^2 + C = 0$, or $Ax^2 + Bx = 0$.

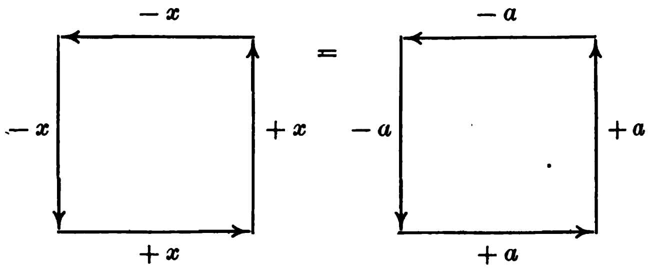
423. The **Fourfold Nature** of a quadratic, whence the name, may be shown by the following methods:

1. Graphic Method

Let $$x^2 = a^2.$$

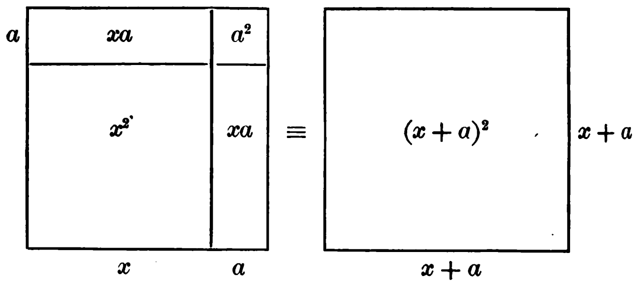
$$\therefore \pm x = \pm a.$$

Since x^2 and a^2 are two equal squares, they may be represented by two equal geometrical squares. x^2 and a^2 represent their respective areas, x and a the length of their respective sides, and the signs + and − the opposition in direction.



This diagram (No. 1) shows the fourfold nature of an *incomplete* quadratic.

The *complete quadratic form* $x^2 + 2xa + a^2 \equiv (x + a)^2$ may be represented by the following diagram, which is made by extending the sides of the square in the preceding diagram.



This diagram (No. 2) shows the fourfold nature of a *complete* quadratic.

Let the student explain diagram 2 and draw another diagram, in which $x = 3$ and $a = 1$. In $x^2 + 2xa + (?)$, what part of the diagram is missing? How can it be found?

2. Algebraic Method

Let $$x^2 = a^2.$$

Then $$\pm x = \pm a.$$

From the last expression it appears that

1. $+x = +a$.
2. $+x = -a$.
3. $-x = -a$.
4. $-x = +a$.

These four equations also show the fourfold nature of an incomplete quadratic.

424. If $x^2 = a^2$, then $x = \pm a$. The question naturally arises, why not put the double sign $\pm$ before x as well as before a after extracting the square root? It is proper to do so, but that it is unnecessary may be shown as follows:

Let $$x^2 = a^2.$$

Then $$\pm x = \pm a.$$

$$\begin{array}{r} 1. \\ 2. \\ 3. \\ 4. \end{array}\left|\begin{array}{l} +x = +a \\ +x = -a \\ -x = -a \\ -x = +a \end{array}\right| = \left|\begin{array}{l} +x = +a \\ +x = -a \\ +x = +a \\ +x = -a \end{array}\right| = \left| x = \pm a \right|$$

Let the student show the same thing by using diagram 1.

425. Problem 1. **To solve a quadratic by factoring.** See Chapter V, Section II, also Prop. 9 in Section I.

Rule and **Dem.** Let the student supply them.

Model Solutions

1. Solve $6x^2 - 5x = 6$.

1. $6x^2 - 5x - 6 = 0$.
2. $(3x + 2)(2x - 3) = 0$.
3. $3x + 2 = 0,\ 2x - 3 = 0$.
4. $3x = -2,\ 2x = +3$.
5. $x = -\frac{2}{3},\ x = +\frac{3}{2}$.

Verification

1. For root $-\frac{2}{3}$.

   1. $\frac{8}{3}+\frac{10}{3}=6$. 2. $6\equiv 6$.

2. For root $+\frac{3}{2}$.

   1. $\frac{27}{2}-\frac{15}{2}=6$. 2. $6\equiv 6$.

METHOD 1, WITH FRACTIONS

2. Solve $3x^2-5x+4=0$ by factoring.

   1. $x^2-\frac{5}{3}x+\frac{25}{36}-\frac{25}{36}+\frac{4}{3}=0$. Why?
   2. $(x-\frac{5}{6})^2-(\frac{25}{36}-\frac{48}{36})=0$. Why?
   3. $(x-\frac{5}{6})^2-(-\frac{23}{36})=0$. Why?
   4. $\{(x-\frac{5}{6})-\frac{1}{6}\sqrt{-23}\}\{(x-\frac{5}{6})+\frac{1}{6}\sqrt{-23}\}=0$.
   5. $x-\frac{5}{6}-\frac{1}{6}\sqrt{-23}=0,\ x-\frac{5}{6}+\frac{1}{6}\sqrt{-23}=0$.
   6. $x=\frac{5}{6}+\frac{1}{6}\sqrt{-23},\ x=\frac{5}{6}-\frac{1}{6}\sqrt{-23}$.
   7. Or $x=\frac{5}{6}\pm\frac{1}{6}\sqrt{-23}$.

METHOD 2, WITHOUT FRACTIONS

3. Solve $3x^2+5x-4=0$ by factoring.

   1. $6x^2+10x-8=0$. Why?
   2. $36x^2+60x-48=0$. Why?
   3. $36x^2+60x+25-25-48=0$. Why?
   4. $(6x+5)^2-73=0$.
   5. $(6x+5-\sqrt{73})(6x+5+\sqrt{73})=0$.
   6. $6x+5-\sqrt{73}=0,\ 6x+5+\sqrt{73}=0$.
   7. $x=-\frac{5}{6}+\frac{1}{6}\sqrt{73},\ x=-\frac{5}{6}-\frac{1}{6}\sqrt{73}$.
   8. Or $x=\dfrac{-5\pm\sqrt{73}}{6}$.

EXAMPLES

Solve by factoring and verify:

1. $x^2 - x + 1 = 0.$
2. $x^2 + 5x + 6 = 0.$
3. $x^2 - 3x + 2 = 0.$
4. $x^2 - 5x - 6 = 0.$
5. $2x^2 - 3x + 5 = 0.$
6. $3x^2 + 4x + 5 = 0.$
7. $3x^2 - 5x - 7 = 0.$
8. $4x^2 - 5x + 6 = 0.$
9. $\frac{2}{3}x^2 - \frac{3}{2}x + \frac{5}{2} = 0.$
10. $\frac{7}{8}x^2 - \frac{5}{6}x - 2 = 0.$
11. $-3x^2 + 2 = -5x.$
12. $x^2 = 4.$
13. $x^2 - 3 = 0.$
14. $x^2 - 81 = 0.$
15. $x^2 + ax = 0.$
16. $\frac{19}{5}x + x^2 = \frac{4}{5}.$
17. $3 = 7x^2 + 5x.$
18. $22x + 23 = x^2.$
19. $3x^2 + 0x - 5 = 0.$
20. $x^2 - (1 - a)x - a = 0.$
21. $x^2 - (a + b)x + ab = 0.$
22. $ax^2 - bcx - a^2x + abc = 0.$
23. $abx^2 + \frac{1}{c}(3a^2x) = \frac{1}{c^2}(6a^2 + ab - 2b^2) - \frac{1}{c}(b^2x).$
24. $\frac{x}{100} + \frac{21}{25x} = \frac{1}{4}.$
25. $4x = \frac{36 - x}{x} + 46.$
26. $\frac{x - 2}{x - 3} + \frac{x - 4}{x - 5} = 3.$
27. $\frac{x - a}{2x} - \frac{x - 3}{x + a} = \frac{1}{2} - \frac{a}{6}.$
28. $5x - \frac{3x - 3}{x - 3} = 2x + \frac{3x - 6}{2}.$
29. $Ax^2 + Bx + C = 0.$
30. $Ax^2 + Bx - C = 0.$
31. $Ax^2 - Bx - C = 0.$
32. $\sqrt{x + 12} = \frac{12}{\sqrt{x + 5}}.$
33. $\frac{3}{x^2 - 3x} + \frac{6}{2x^2 + 8x} = \frac{27}{8x}.$
34. $\sqrt{3x - 5} = \frac{1}{5}\sqrt{7x^2 + 15x}.$
35. $\sqrt{(4 + x)(5 - x)} = 2x - 10.$

426. PROBLEM 2. **To solve a quadratic equation by completing the square, not avoiding fractions.**

Rule. *Reduce the equation to the form* $x^2 + bx = c$. *Divide the second term of the first member by twice the square root of*

the first term, and add the square of the result to both members. Extract the square root of both members, and find the value of x in the usual way.

Dem. Since the second term of the trinomial square equals twice the product of the square roots of the first and the third terms, the third term can always be found, if missing, by dividing the second by twice the square root of the first and squaring the result.

Model Solutions

1. Solve $3x^2 - x - 10 = 0$.

1. $3x^2 - x = 10$.
2. $x^2 - \frac{1}{3}x = \frac{10}{3}$.
3. $x^2 - \frac{1}{3}x + \frac{1}{36} = \frac{10}{3} + \frac{1}{36}$.
4. $(x - \frac{1}{6})^2 = \frac{121}{36}$.
5. $x - \frac{1}{6} = \pm\frac{11}{6}$.
6. $x = \frac{1}{6} \pm \frac{11}{6}$.
7. $x = 2$, or $-\frac{5}{3}$.

2. Solve $5x^2 - 3x + 7 = 0$.

1. $x^2 - \frac{3}{5}x = -\frac{7}{5}$.
2. $x^2 - \frac{3}{5}x + \frac{9}{100} = -\frac{140}{100} + \frac{9}{100}$.
3. $x^2 - \frac{3}{5}x + \frac{9}{100} = -\frac{131}{100}$.
4. $x - \frac{3}{10} = \pm\sqrt{-\frac{131}{100}}$.
5. $x = \frac{1}{10}(3 \pm \sqrt{-131})$.

EXAMPLES

Solve and verify:

1. $5x^2 + 13x - 6 = 0$.
2. $5x^2 + 17x = -6$.
3. $5x^2 - 17x - 6 = 0$.
4. $5x^2 - 13x - 6 = 0$.
5. $cx^2 - 0x + 1 = 0$.
6. $2x^2 + 3x + 4 = 0$.
7. $3x^2 - 4x - 5 = 0$.
8. $4x^2 + 5x - 0 = 0$.
9. $5z^2 - 6z - 7 = 0$.
10. $6s^2 + 7s - 8 = 0$.
11. $7t^2 - 8t + 9 = 0$.
12. $Q - \frac{5}{7} + \frac{2}{3}Q^2 = 4$.
13. $\frac{2}{3}v - \frac{3}{2}v^2 - 1 = 2$.
14. $-8y^2 - 9y + 10 = 0$.
15. $\dfrac{3}{5 - x} = 2 + \dfrac{1}{2x - 5}$.
16. $\sqrt{3x + 4} + \sqrt{3x - 5} = 9$.

17. $\frac{1}{x^2-1}-\frac{1}{1-x}=\frac{7}{8}-\frac{1}{x+1}.$

18. $\frac{1}{x^2-1}-\frac{1}{x+1}=\frac{1}{3(x-1)}-\frac{1}{3}.$

19. $\sqrt{2x+9}=\sqrt{x-4}+\sqrt{x+1}.$

20. $x^2+(a+b+2c)x+(a+b+c)x=0.$

427. PROBLEM 3. **To solve a quadratic equation by completing the square, avoiding fractions.**

Rule. *Reduce the equation to the form $ax^2+bx=c$. Multiply both members by a number that will make the second term exactly divisible by twice the square root of the first term, and then proceed as in Problem 2.*

Dem. Let the student state the reasons for each step.

MODEL SOLUTIONS

1. Solve $5x^2+3x=-7$.

1. $10x^2+6x=-14.$
2. $100x^2+60x=-140.$
3. $100x^2+60x+9=-140+9.$
4. $(10x+3)^2=-131.$
5. $10x+3=\pm\sqrt{-131}.$
6. $10x=-3\pm\sqrt{-131}.$
7. $x=\frac{-3\pm\sqrt{-131}}{10}.$

2. Solve $Ax^2+Bx+C=0$.

1. $4A^2x^2+4ABx+4AC=0.$
2. $4A^2x^2+4ABx+B^2=B^2-4AC.$
3. $2Ax+B=\pm\sqrt{B^2-4AC}.$
4. $2Ax=-B\pm\sqrt{B^2-4AC}.$
5. $x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$

EXAMPLES

Solve and verify:

1. $6x^2 - 7x = 3.$
2. $4x^2 - 3x = 0.$
3. $\frac{7}{9}x^2 - 2\frac{1}{2} = x.$
4. $\frac{12}{5}x = \frac{4}{5} - x^2.$
5. $-\frac{7}{6}x + x^2 = \frac{1}{2}.$
6. $\frac{3}{5} - x^2 + 7x = 2.$
7. $ax^2 - bx + c = 0.$
8. $ax^2 - bx - c = 0.$
9. $ax^2 + bx - c = 0.$
10. $5x^2 - 0x + 2 = 0.$
11. $6x^2 - 11x + 3 = 0.$
12. $3x^2 - 5x + 11 = 0.$
13. $acx^2 - bdx + fg = 0.$
14. $x + \frac{1}{a} = a + \frac{1}{x}.$
15. $ax^2 + bx - \frac{b}{2} = \frac{a}{4}.$
16. $\frac{x+3}{x-4} + \frac{x-2}{x+3} = 4.$
17. $\frac{x+3}{2x-7} - \frac{2x-1}{x-3} = 0.$
18. $\frac{\frac{3}{2}x+2}{\frac{3}{2}x-1} - \frac{7}{3} = \frac{4-\frac{3}{2}x}{3x}.$
19. $\frac{x-a}{2x} - \frac{x-3a}{x+a} = \frac{1}{2} - \frac{a}{6}.$
20. $2(x-\frac{1}{2}) - 4(x+\frac{1}{2})^2 + 134 = 0.$
21. $10x^2 + (a+11)x + b + 12 - 0.$
22. $\frac{3x - \sqrt{x^2-8}}{x - \sqrt{x^2-8}} = x + \sqrt{x^2-8}.$
23. $\frac{1}{1+\sqrt{1-x}} + \frac{1}{1-\sqrt{1-x}} = \frac{3}{7}x + 5.$

428. PROBLEM 4. **To solve a quadratic equation by the use of a formula.**

Rule. *Reduce the equation to the form* $Ax^2 + Bx + C = 0$, *and substitute in the formula* $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ *the corresponding values in any particular example.*

Dem. Since $Ax^2 + Bx + C = 0$ is the type of all quadratic equations, and $\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ are its roots (see model solution No. 2, Problem 3), the substitution of the corresponding values of A, B, and C in any particular equation will give the roots of that equation.

429. The expression $B^2 - 4AC$ in the foregoing formula is called the **Discriminant** of the equation $Ax^2 + Bx + C = 0$.

Model Solutions

1. Solve $12x^2 - 11x - 15 = 0$.

Formula, $$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

In this case $A = 12,\ B = -11,\ C = -15.$

$$\therefore x = \frac{11 \pm \sqrt{121 + 720}}{24} = \frac{11 \pm \sqrt{841}}{24} = \frac{11 \pm 29}{24} = \frac{5}{3} \text{ and } -\frac{3}{4}.$$

2. Solve $13x^2 - 5x + 2 = 0$.

$$x = \frac{5 \pm \sqrt{25 - 104}}{26} = \frac{5 \pm \sqrt{-79}}{26}.$$

EXAMPLES

Solve by the formula and verify:

1. $ax^2 + bx - c = 0.$
2. $x^2 + 8x + 14 = 0.$
3. $\frac{x^2}{81} - \frac{4x}{18} + 5 = 0.$
4. $3x^2 + x - 11 = 0.$
5. $3x - \frac{x^2}{4} + 3 = 13.$
6. $3x^2 - 2x - 65 = 0.$
7. $x^2 - 4x - 55 = 2x.$
8. $\frac{7+x}{7-x} + \frac{7-x}{7+x} = \frac{29}{10}.$
9. $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}.$
10. $\frac{4}{3}x^2 - x = \frac{85}{3}.$
11. $x^{-2} + x^{-1} = 6.$
12. $\frac{a^2}{b^2}x^2 - \frac{2a}{c}x + \frac{b^2}{c^2} = 0.$
13. $8x^2 - 7x + 34 = 0.$
14. $6x + \frac{1}{x}(35 - 3x) = 44.$
15. $\frac{1}{64}x^2 + x + 16 = 16x^2.$
16. $\frac{1}{9}x^2 + \frac{1}{3}x - 9 = -\frac{2\frac{2}{9}}{4}.$
17. $mqx^2 - mnx + pqx = np.$
18. $\frac{1}{2}(8-x) - \frac{2x-11}{x-3} = \frac{x-2}{6}.$

Section II

DISCUSSION OF ROOTS

430. The **Two Roots** of a Quadratic Equation may both be positive, both negative, or one positive and one negative. They are real or imaginary; equal or unequal; rational or irrational.

Questions for Discussion

1. Are the roots real or imaginary, and why?
2. Are the roots equal or unequal, and why?
3. Are the roots rational or irrational, and why?
4. Are the roots positive, negative, or both, and why?
5. If both positive and negative, which is the greater, and why?

Regular Cases, B and $C \neq 0$

1. When B is $+$ and C is $+$.
2. When B is $-$ and C is $+$.
3. When B is $+$ and C is $-$.
4. When B is $-$ and C is $-$.

Conditions

1. When $B^2 - 4AC > 0$, *i.e.*, positive.
2. When $B^2 - 4AC = 0$.
3. When $B^2 - 4AC < 0$, *i.e.*, negative.

Case 1. B is $+$, C is $+$

Equation, $$Ax^2 + Bx + C = 0.$$

Roots, $$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Discussion

$$1.\ B^2-4AC>0\begin{cases}\text{1. Roots are real} \because B^2-4AC \text{ is } +. \quad \text{Why?}\\ \text{2. Roots are unequal} \because \sqrt{\ } \neq 0. \quad \text{Why?}\\ \text{3. Roots are rational if } B^2-4AC \text{ is a perfect square.}\\ \text{4. Roots are both } - \because -B>\sqrt{B^2-4AC}. \quad \text{Why?}\end{cases}$$

$$2.\ B^2-4AC=0\begin{cases}\text{1. Roots are real} \because \sqrt{\ }=0. \quad \text{Why?}\\ \text{2. Roots are equal} \because \sqrt{\ }=\pm 0. \quad \text{Why?}\\ \text{3. Roots are rational} \because \sqrt{\ }=0.\\ \text{4. Roots are both } - \because B \text{ is } -.\end{cases}$$

$$3.\ B^2-4AC<0\begin{cases}\text{1. Roots are imaginary} \because B^2-4AC \text{ is } -.\\ \text{2. Roots are unequal} \because \sqrt{\ } \neq 0.\\ \text{3. Roots are irrational} \because B^2-4AC \neq \text{perfect square.}\end{cases}$$

Case 2. B is $-$, C is $+$

Equation, $$Ax^2-Bx+C=0.$$

Roots, $$x=\frac{+B\pm\sqrt{B^2-4AC}}{2A}.$$

Discussion

Let the student write out the discussion, following the form used in discussing Case 1.

Case 3. B is $+$, C is $-$

Equation, $$Ax^2+Bx-C=0.$$

Roots, $$x=\frac{-B\pm\sqrt{B^2+4AC}}{2A}.$$

Discussion

$$B^2+4AC>0\begin{cases}\text{1. Roots are real} \because B^2+4AC \text{ is } +.\\ \text{2. Roots are unequal} \because \sqrt{\ } \neq 0.\\ \text{3. Roots are rational if } B^2+4AC \text{ is a perfect square.}\\ \text{4. Roots are } + \text{ and } - \because B<\sqrt{B^2+4AC}. \quad \text{Why?}\\ \text{5. } - \text{root} > + \because \text{numerical sum} > \text{numerical difference.}\end{cases}$$

Case 4. B is $-$, C is $-$

Equation, $Ax^2 - Bx - C = 0.$

Roots, $$x = \frac{B \pm \sqrt{B^2 + 4AC}}{2A}.$$

Discussion

Let the student write out the discussion, following the form used in discussing Case 3.

Special Cases

1. When $B = 0$, $C \neq 0$. 2. When $B \neq 0$, $C = 0$.
3. When B and $C = 0$.

Case 1. $B = 0$, $C \neq 0$

Equation, $Ax^2 + 0x + C = 0.$

Roots, $$x = \frac{0 \pm \sqrt{0^2 - 4AC}}{2A} = \pm \frac{1}{A}\sqrt{-AC}.$$

Discussion

1. The roots are numerically equal. Why?
2. The roots are positive and negative. Why?
3. The roots are real or imaginary, rational or irrational, depending upon $\sqrt{-AC}$. Why?

Case 2. $B \neq 0$, $C = 0$

Equation, $$Ax^2 + Bx + 0 = 0,$$
$$(Ax + B)x = 0.$$

Roots, $x = -\frac{B}{A}$ and 0. Why?

Discussion

1. The roots are real. Why?
2. The roots are unequal. Why?
3. The roots are rational. Why?
4. One root is always 0. Why?

Case 3. $B = 0,\ C = 0$

Equation, $$Ax^2 + 0x + 0 = 0.$$

$$Ax^2 = 0.$$

Roots, $x = \pm 0.$ Why?

Discussion

1. The roots are real.
2. The roots are numerically equal.
3. The roots are positive and negative. Why?
4. The roots are rational.

EXAMPLES

Discuss the roots of $3x^2 + 7x + 2 = 0.$

Model Discussion

Roots, $$x = \frac{-7 \pm \sqrt{49 - 24}}{6}.$$

Case 1. *Condition*, $B^2 - 4AC > 0$

1. Roots are real, $\because$ $49 - 24$ is $+$.
2. Roots are unequal, $\because$ $\sqrt{25} \neq 0$.
3. Roots are rational, $\because$ $49 - 24$ is a perfect square.
4. Roots are both $-$, $\because$ $\sqrt{25} < 7$, and 7 is negative.

Discuss the roots of the following:

1. $x^2 + 5x + 6 = 0.$
2. $x^2 - 5x + 6 = 0.$
3. $x^2 - x - 6 = 0.$
4. $x^2 + x - 6 = 0.$
5. $2x^2 - 3x + 5 = 0.$
6. $2x^2 + 3x + 5 = 0.$
7. $2x^2 - 3x - 5 = 0.$
8. $2x^2 + 3x - 7 = 0.$
9. $2x^2 - 72 = 0.$
10. $4x^2 - 5x = 0.$

431. The **relations between the coefficients and the roots** of a quadratic equation may be shown as follows:

Let $$Ax^2 + Bx + C = 0, \text{ or}$$

$$x^2 + \frac{B}{A}x + \frac{C}{A} = 0.$$

Then

1. The *sum* of the roots is equal to the coefficient of x, with its sign changed, divided by the coefficient of x^2, or $-\frac{B}{A}$.

2. The *product* of the roots is equal to the known term divided by the coefficient of x^2, or $\frac{C}{A}$.

Dem. If $Ax^2 + Bx + C = 0$,

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \text{ and } \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

Then

1. $$\frac{-B + \sqrt{B^2 - 4AC}}{2A} + \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

$$= -\frac{2B}{2A} = -\frac{B}{A}.$$

2. $$\frac{-B + \sqrt{B^2 - 4AC}}{2A} \cdot \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{B^2 - B^2 + 4AC}{4A^2} = \frac{C}{A}.$$

MODEL SOLUTIONS

1. What is (1) the sum of the roots, (2) the product of the roots, of the following equations?

1. $x^2 + 2x + 3 = 0$,
2. $x^2 - 3x - 2 = 0$,
3. $x^2 + 7x + 4 = 0$,
4. $x^2 - x + 5 = 0$.

1. Sum $= -2 + 3 - 7 + 1 = -5$. Why?
2. Product $= 3 \cdot -2 \cdot 4 \cdot 5 = -120$. Why?

2. What is (1) the sum of the roots, (2) the product of the roots, of the equation $3x^2 + 5x + 7 = 0$?

1. Sum $= -\frac{5}{3}$;
2. Product $= +\frac{7}{3}$. Why?

3. Form an equation whose roots are $\frac{2}{3}$ and $-\frac{4}{5}$.

1. Sum $= \frac{2}{3} - \frac{4}{5} = -\frac{2}{15}$;
2. Product $= \frac{2}{3} \cdot -\frac{4}{5} = -\frac{8}{15}$.
3. Equation is $x^2 + \frac{2}{15}x - \frac{8}{15} = 0$, or
4. $15x^2 + 2x - 8 = 0$.

4. If m and n are the roots of $x^2 + px + q = 0$, find the equation whose roots are $m + n$ and $\frac{1}{m} + \frac{1}{n}$.

1. $m + n = -p$. Why?
2. $mn = q$. Why?
3. $\frac{1}{m} + \frac{1}{n} = \frac{m+n}{mn} = \frac{-p}{q}$. Why?
4. Sum $= m + n + \frac{1}{m} + \frac{1}{n} = -p + \frac{-p}{q} = -\frac{qp+p}{q}$.
5. Product $= (m+n)\left(\frac{1}{m} + \frac{1}{n}\right) = (-p)\left(-\frac{p}{q}\right) = \frac{p^2}{q}$.
6. Equation is $x^2 + \frac{qp+p}{q}x + \frac{p^2}{q} = 0$, or
7. $qx^2 + (pq + p)x + p^2 = 0$.

5. What must be the value of k that $3x^2 + 8x + k = 0$ may have equal roots?

$$\text{If } 3x^2 + 8x + k = 0,$$

$$x = \frac{-8 \pm \sqrt{64 - 12k}}{6}.$$

The condition that this equation shall have equal roots is that

$$64 - 12k = 0. \quad \text{Why?}$$

$$-12k = -64.$$

$$\therefore k = 5\tfrac{1}{3}.$$

EXAMPLES

Form equations whose roots are

1. $2, -3$.
2. m, n.
3. $\frac{m}{n}, \frac{n}{m}$.
4. $m + n, \frac{1}{m} + \frac{1}{n}$.
5. $\frac{p+q}{p}, \frac{p-q}{q}$.
6. $s + \frac{1}{t}, t + \frac{1}{s}$.
7. $a + b\sqrt{-1}, a - b\sqrt{-1}$.

If m and n are the roots of $Ax^2 + Bx + C = 0$,

8. Show that $\frac{m+n}{mn} = -\frac{B}{C}$.

9. Show that $m^2 + n^2 = \frac{B^2}{A^2} - \frac{2C}{A}$.

10. Show that $\frac{m+n}{m-n} = -\frac{B\sqrt{B^2 - 4AC}}{B^2 - 4AC}$.

11. Show that $\frac{1}{m^3} + \frac{1}{n^3} = \frac{3AB}{C^2} - \frac{B^3}{C^3}$.

12. Show that mn and $m + n$ are the roots of the equation

$$A^2x^2 + (AB - AC)x - BC = 0.$$

13. Show that $\frac{m}{n}$ and $\frac{n}{m}$ are the roots of

$$ACx^2-(B^2-2AC)x+AC=0.$$

14. What must be the value of k that $x^2-10x+k^2=0$ shall have equal roots?

15. What must be the value of k that $5x^2+2x+k=0$ shall have equal roots?

16. If m and n are the roots of $x^2+5x-50=0$, find the equation whose roots are $\frac{m}{n}$ and $\frac{n}{m}$.

17. If m and n are the roots of $Ax^2+px+q=0$, find the equation whose roots are $m+n$ and $\frac{1}{m}+\frac{1}{n}$.

18. Show that the roots of $x^2-5x+6=0$ are the cube roots of $x^2-35x+216=0$.

MISCELLANEOUS EXAMPLES

Solve the following examples by any method:

1. $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\dfrac{1}{a}$.

2. $9a^4b^4x^2-6a^3b^2x=b^2$.

3. $\dfrac{ab}{a^2-b^2x^2}+\dfrac{cd}{c^2-d^2x^2}=0$.

4. $x^2-ax+b=ax(x-1)$.

5. $\dfrac{\sqrt{a+1}}{\sqrt{x-a}+\sqrt{ax-1}}=\dfrac{1}{\sqrt{x-1}}$.

6. $(7-4\sqrt{3})x^2+(2-\sqrt{3})x=2$.

7. $\dfrac{x+m-2n}{x+m+2n}=\dfrac{n+2m-2x}{n-2m+2x}$.

8. $\dfrac{x+\sqrt{x}}{x-\sqrt{x}}=\dfrac{3\sqrt{x}+6}{2\sqrt{x}}$.

9. $\dfrac{x+1}{c}-\dfrac{2}{cx}=\dfrac{x+2}{ax-bx}$.

10. $(a+b)x^2=cx+\dfrac{ac}{a+b}$.

11. $\dfrac{\sqrt{a^2-x^2}-\sqrt{b^2+x^2}}{\sqrt{a^2-x^2}+\sqrt{b^2+x^2}}=\dfrac{c}{d}$.

12. $\sqrt[3]{\dfrac{1}{1+x}\sqrt{\dfrac{1}{1+x}}}=\frac{1}{12}\sqrt{2x}$.

13. $(ax-b)^2-4a(ax-b)=\frac{9}{4}a^2$.

14. $\dfrac{a+b-x^2}{3a-b-3x^2}=\dfrac{3(a-b+x^2)}{a-5b+x^2}.$

15. $2\sqrt{x-a}+3\sqrt{2x}=\dfrac{7a+5x}{\sqrt{x-a}}.$

16. $\dfrac{5a-6b+x}{a+x}=\dfrac{3a-5b+3x}{a+b+x}.$

17. $\dfrac{\sqrt{3}}{\sqrt{2x-1}-\sqrt{x-2}}=\dfrac{1}{\sqrt{x-1}}.$

18. $\dfrac{(a+2b)x}{a-2b}=\dfrac{a^2}{a-2b}-\dfrac{4b^2}{x}.$

19. $\dfrac{2x+\sqrt{x}}{2x-\sqrt{x}}=3\tfrac{7}{15}-3\cdot\dfrac{2x-\sqrt{x}}{2x+\sqrt{x}}.$

20. $\sqrt{1+x}+\sqrt{1+x+\sqrt{1-x}}=1.$

21. $\dfrac{x+m}{x-m}+\dfrac{x-m}{x+m}=\dfrac{2(m^2+1)}{(1+m)(1-m)}.$

22. $\sqrt{a+x}+\sqrt{b+x}=\sqrt{a+b+2x}.$

23. $\sqrt{x-3}+\sqrt{3x+4}+\sqrt{x+2}=0.$

24. $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}.$

25. $\dfrac{\sqrt{x^2-a+b}-\sqrt{x^2+a-b}}{\sqrt{x^2-a+b}+\sqrt{x^2+a-b}}=\dfrac{a-b}{a+b}.$

26. $\dfrac{x-6}{\sqrt{x}-\sqrt{6}}+\dfrac{\sqrt{x-6}}{\sqrt{x}}=\sqrt{x}-\dfrac{\sqrt{x}}{\sqrt{x-6}}.$

27. $\dfrac{1}{\sqrt{a-x}+\sqrt{a}}+\dfrac{1}{\sqrt{a+x}-\sqrt{a}}=\dfrac{\sqrt{a}}{x}.$

28. $\sqrt{3a^2+\sqrt{x}}-(3a^2-x^{\frac{1}{2}})^{\frac{1}{2}}=2a^{\frac{1}{2}}x^{\frac{1}{4}}\sqrt{b^{-1}}.$

29. $\dfrac{a}{b(2x-1)}-\dfrac{b(2x+1)}{a(x^2-1)}=\dfrac{1}{(2x-1)(x+1)}.$

30. $\sqrt{x^2-3x+2}+\sqrt{(x-3)(x-4)}-\sqrt{2}=0.$

31. $(3+b^2)(x^2-x+1)=(3b^2+1)(x^2+x+1).$

32. $(4a^2-9cd^2)x^2+(4a^2c^2+4abd^2)x+(ac^2+bd^2)^2=0.$

SECTION III

HIGHER EQUATIONS SOLVED AS QUADRATICS

432. Prop. 1. Any equation which can be reduced to the form $Ax^{2n} + Bx^n + C = 0$ can be solved, partially at least, as a quadratic.

Dem. Since $Ax^{2n} + Bx^n + C = 0$ is a quadratic in x^n, it may be solved for x^n by one of the usual methods. Solving for x^n by the formula

$$x^n = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

from which at least two values of x may be obtained by extracting the nth root of both members.

$$\therefore\ x = \sqrt[n]{\frac{-B + \sqrt{B^2 - 4AC}}{2A}} \text{ and } \sqrt[n]{\frac{-B - \sqrt{B^2 - 4AC}}{2A}}.$$

If n is even, how many roots at least can be obtained? Why? Why is $\pm$ not written before $\sqrt[n]{\ }$?

433. PROBLEM 1. **To solve an equation of the form**

$$Ax^{2n} + Bx^n + C = 0.$$

Rule. Let the student make the Rule and the Demonstration from the Proposition.

MODEL SOLUTIONS

1. Solve $x^6 - 19x^3 - 216 = 0$.

1. $(x^3 + 8)(x^3 - 27) = 0.$
2. $(x + 2)(x^2 - 2x + 4)(x - 3)(x^2 + 3x + 9) = 0.$
3. $x + 2 = 0.$ Why?
4. $x^2 - 2x + 4 = 0.$ Why?
5. $x - 3 = 0.$ Why?

6. $x^2 + 3x + 9 = 0.$ **Why?**

7. $x = -2.$

8. $x = 1 \pm \sqrt{-3}.$

9. $x = 3.$

10. $x = \frac{1}{2}(-3 \pm 3\sqrt{-3}).$

How many roots are found? Is this the whole number? Why? How many of the roots are imaginary? How many are real?

State the propositions in Factoring used in this example. What method of solving a quadratic will give all the roots of an equation? Is it always practicable?

2. Solve $x^6 - 19x^3 - 216 = 0.$

$$x^3 = \frac{19 \pm \sqrt{19^2 + 864}}{2} = 27 \text{ and } -8, \text{ by formula.}$$

$$x = 3 \text{ and } -2, \text{ by extracting cube root.}$$

How many roots are lost by this method of solution?
Can six roots be found from $x^3 = 27$ and -8? How? Find them.

3. Solve $\sqrt{x^2 + 12} + \sqrt[4]{x^2 + 12} = 6.$

1. $(\sqrt[4]{x^2 + 12} - 2)(\sqrt[4]{x^2 + 12} + 3) = 0.$ **Why?**
2. $\sqrt[4]{x^2 + 12} - 2 = 0.$ **Why?**
3. $\sqrt[4]{x^2 + 12} + 3 = 0.$ **Why?**
4. $\sqrt[4]{x^2 + 12} = 2$ and $-3.$
5. $x^2 + 12 = 16$ and $81.$
6. $x^2 = 4$ and $69.$
7. $x = \pm 2$ and $\pm\sqrt{69}.$

4. Solve $2x^2 - 6x + 14\sqrt{11x - 2x^2 + 2} = 5x + 42.$

1. $2x^2 - 11x + 14\sqrt{11x - 2x^2 + 2} - 42 = 0.$
2. $-2x^2 + 11x + 2 - 14\sqrt{-2x^2 + 11x + 2} + 40 = 0.$ **Why?**
3. $y^2 - 14y + 40 = 0.$ **Why?**
4. $(y - 4)(y - 10) = 0.$

5. $y = 4$ and 10.

6. $\sqrt{-2x^2 + 11x + 2} = 4$ and 10. Why?

7. $-2x^2 + 11x + 2 = 16$ and 100. Why?

8. $2x^2 - 11x + 14 = 0$ and $2x^2 - 11x + 98 = 0$.

9. $(x - 2)(2x - 7) = 0$ and $(x - \frac{11}{4} \mp \frac{1}{4}\sqrt{-663}) = 0$.

10. $x = 2, \frac{7}{2}, \frac{11}{4} \pm \frac{1}{4}\sqrt{-663}$.

5. Solve $\sqrt{2x} - 5x = 5$.

1. $5x - \sqrt{2} \cdot \sqrt{x} + 5 = 0$. Why?

2. $\sqrt{x} = \dfrac{\sqrt{2} \pm \sqrt{2 - 100}}{10}$.

3. $x = \dfrac{2 \pm 2\sqrt{-196} - 98}{100}$. Why?

4. $x = -\frac{24}{25} \pm \frac{7}{25}i$. Why?

EXAMPLES

Find the roots of:

1. $x^4 - \sqrt{x^4} = 20$.

2. $4x^{\frac{8}{3}} + x^{\frac{4}{3}} = 39$.

3. $\dfrac{x}{2} - \dfrac{1}{3}\sqrt{x} = 22\frac{1}{6}$.

4. $3x^3 + 2\sqrt{x^3} = 1$.

5. $z^{2x} + 4z^x - 5 = 0$.

6. $3x^{\frac{4}{3}} - \frac{5}{2}x^{\frac{8}{3}} = 592$.

7. $2\sqrt[3]{x^2} + 3\sqrt[3]{x} = 2$.

8. $\dfrac{x}{a + x} + \dfrac{a}{\sqrt{a + x}} = \dfrac{b}{x}$.

9. $\frac{3}{5}\sqrt{x} - 2 = \frac{1}{20}(x - 5)$.

10. $x^{\frac{2}{n}} - 5\sqrt[n]{x} + 6 = 0$.

11. $\dfrac{x}{x + 4} + \dfrac{4}{\sqrt{x + 4}} = \dfrac{21}{x}$.

12. $\left(x + \dfrac{1}{x}\right)^2 + \left(x + \dfrac{1}{x}\right) = 1$.

13. $2x^{-\frac{4}{3}} - 21x^{-\frac{2}{3}} + 27 = 0$.

14. $x^2 - 16 + \sqrt{x^2 - 16} = 12$.

15. $\dfrac{7 - 12x^2}{\sqrt{x^3}} = \dfrac{x^2}{\sqrt{x}} - \dfrac{8x^2 + 110}{\sqrt{x^3}}$.

16. $y + 4 + \sqrt{\dfrac{y + 4}{y - 4}} - \dfrac{12}{y - 4} = 0$.

17. $x+\sqrt{x^2-ax+b^2}-a^{-1}x^2=b.$

18. $(x-1)^{-\frac{4}{3}}-5(x-1)^{-\frac{2}{3}}+1=0.$

19. $x^2-2x+6\sqrt{x^2-2x+5}=11.$

20. $\left(\dfrac{x^2+1}{x}\right)^2+4\left(\dfrac{x^2+1}{x}\right)-12=0.$

21. $6x^2-2x-3\sqrt{3x^2-x+1}=3.$

22. $\dfrac{2}{\sqrt{(x+2)^3}}+\dfrac{\sqrt{x+2}}{2}=\dfrac{17}{4(x+2)^{\frac{1}{2}}}.$

23. $9x-4x^2+\sqrt{4x^2-9x+11}=5.$

24. $(x^2-5x)^2-8(x^2-5x)-84=0.$

25. $5x-7x^2-8\sqrt{7x^2-5x+1}=8.$

26. $(2x-3)^{-6}+7(2x-3)^{-3}-8=0.$

27. $3x(3-x)=11-4\sqrt{x^2-3x+5}.$

28. $2x\sqrt{x-a}+3\sqrt{2x}=\dfrac{5x^2-5ax}{\sqrt{x-a}}.$

29. $x^2-x+5\sqrt{2x^2-5x+6}=\dfrac{3x+33}{2}.$

30. $x^2-10x-2\sqrt{x^2-10x+18}=-15.$

31. $2x^2-2x+2\sqrt{2x^2-7x+6}=5x-6.$

32. $6(x^2-2x+5)+36\sqrt{x^2-2x+5}=96.$

434. Prop. 2. Any equation which can be reduced to the form $(x-a)(x-b)(x-c)\cdots=0$ may be solved by putting each factor equal to zero.

Dem. See Chapter V, Section II.

435. Problem 2. Let the student make the Problem and its Rule from the following

Model Solutions

1. Solve $x^5 - 2x^4 - 15x^3 + 8x^2 + 68x + 48 = 0.$

$$\begin{array}{l}
1 - 2 - 15 + 8 + 68 + 48 \,\underline{|+1} \\
\quad +1 - 3 - 12 + 20 + 48 \\
\hline
1 - 3 - 12 + 20 + 48 \,\underline{|+2} \\
\quad +2 - 10 - 4 + 48 \\
\hline
1 - 5 - 2 + 24 \,\underline{|-3} \\
\quad -3 + 6 + 24 \\
\hline
1 - 2 - 8 \,\underline{|+2} \\
\quad +2 - 8 \\
\hline
1 - 4 \,\underline{|-4} \\
\quad -4 \\
\hline
1
\end{array}$$

$\therefore \qquad (x+1)(x+2)(x-3)(x+2)(x-4) = 0$, and

$$x = -1,\ -2,\ +3,\ -2,\ +4.$$

2. Solve $x^4 - 2x^3 + 3x^2 - 2x - 3 = 0.$

1. $(x^2 - x + 3)(x^2 - x - 1) = 0$, by Prop. 8 in Factoring.
2. $\therefore\ x = \dfrac{1 \pm \sqrt{-11}}{2}$ and $\dfrac{1 \pm \sqrt{5}}{2}$, by formula.

3. $x - 3 = \dfrac{3 + 4\sqrt{x}}{x}.$

1. $x^2 - 3x = 3 + 4\sqrt{x}.$
2. $x^{\frac{4}{2}} + 0x^{\frac{3}{2}} - 3x^{\frac{2}{2}} - 4x^{\frac{1}{2}} - 3 = 0.$
3. $\begin{cases} x^{\frac{2}{2}} + x^{\frac{1}{2}} + 1 = 0. \\ x^{\frac{2}{2}} - x^{\frac{1}{2}} - 3 = 0. \end{cases}$ (3, 4)
5. $x = \left(\dfrac{-1 \pm \sqrt{-3}}{2}\right)^2$ and $\left(\dfrac{1 \pm \sqrt{13}}{2}\right)^2.$

EXAMPLES

Solve and verify:

1. $x^3 - 1 = 0.$

2. $x^3 - 8x - 3 = 0.$

3. $\dfrac{v}{4} - \dfrac{\sqrt{v} - 12}{v - 18} = 0.$

4. $\dfrac{t + \sqrt{t}}{t - \sqrt{t}} - \dfrac{t^2 - t}{9} = 0.$

5. $x^2 + \dfrac{1}{x^2} - a^2 - \dfrac{1}{a^2} = 0.$

6. $x^3 - \frac{13}{12}x^2 + \frac{3}{8}x - \frac{1}{24} = 0.$

7. $x^{\frac{5}{2}} + x^{\frac{3}{2}} - 6x^{\frac{1}{2}} = 0.$

8. $x^3 - 7x^2 + 16x - 12 = 0.$

9. $\sqrt{Q} - \dfrac{8}{Q} - \dfrac{7}{\sqrt{Q} - 2} = 0.$

10. $x^3 + \frac{17}{4}x^2 - 9\frac{7}{8}x + 3\frac{3}{4} = 0.$

11. $x + \dfrac{1}{x} + \dfrac{7}{2} \cdot \dfrac{x+1}{\sqrt{x}} - 13 = 0.$

12. $x + a - b + 3a^{\frac{1}{3}}b^{\frac{1}{3}}x^{\frac{1}{3}} = 0.$

13. $(x-4)^2 + 2(x-4) - \dfrac{2}{x} + 1 = 0.$

14. $x^4 - 3x^3 - 14x^2 + 48x - 32 = 0.$

15. $\dfrac{\sqrt{x+m}}{\sqrt{x+m} + \sqrt{m}} = \dfrac{\sqrt{m-x}}{\sqrt{m} - \sqrt{m-x}}.$

16. $(1 + x + x^2)^2 = \dfrac{a+1}{a-1}(x^4 + x^2 + 1).$

17. $4x^4 + 16x^3 - 57x^2 + 16x + 4 = 0.$

18. $x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2 = 0.$

19. $x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0.$

20. $3\sqrt{x^3 + 17} + \sqrt{x^3 + 1} + 2\sqrt{5x^3 + 41} = 0.$

436. A **Reciprocal Equation** is of the form

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Cx^2 + Bx + Ax^0 = 0.$$

The coefficients of the terms of the polynomial equidistant from the first and the last terms respectively are numerically

equal, the signs being all alike, or all opposite; and when $\frac{1}{x}$ is substituted for x the new equation is equivalent to the first.

ILLUSTRATIONS.

1. $x^5+5x^4+10x^3+10x^2+5x+1=0$, signs all alike.
2. $x^3-3x^2+3x-1=0$, signs opposite.
3. $\frac{1}{x^3}-\frac{3}{x^2}+\frac{3}{x}-1=0$, $\frac{1}{x}$ substituted for x.
4. $Ax^2+Bx+A=0$.

437. PROBLEM 3. **To solve a reciprocal equation.**

Rule 1. *Reduce the equation to the form*

$$Ax^n+Bx^{n-1}+\cdots+Bx+A=0.$$

Then, when possible, solve by the method of factoring, or

Rule 2. *Divide by some power of x so that the polynomial may be grouped in the form of a quadratic.*

Dem. See Problems 1 and 2.

MODEL SOLUTIONS

1. Solve $x^3+x^2+x+1=0$.

   1. $x^2(x+1)+(x+1)=0$.
   2. $(x+1)(x^2+1)=0$.
   3. $x=-1, \pm i$.

2. Solve $x^4-x^3+x^2-x+1=0$.

   1. $x^2-x+1-\frac{1}{x}+\frac{1}{x^2}=0$, by dividing by x^2.
   2. $\left(x^2+\frac{1}{x^2}\right)-\left(x+\frac{1}{x}\right)+1=0$, by grouping.
   3. $\left(x^2+2+\frac{1}{x^2}\right)-\left(x+\frac{1}{x}\right)-1=0$. Why?

4. $y^2 - y - 1 = 0.$ Why?

5. $y = \frac{1}{2}(1 \pm \sqrt{5}).$

6. $x + \dfrac{1}{x} = \dfrac{1}{2}(1 \pm \sqrt{5}).$ Why?

7. $2x^2 - (1 \pm \sqrt{5})x + 2 = 0.$

8. $x = \dfrac{1 \pm \sqrt{5} \pm \sqrt{(1 \pm \sqrt{5})^2 - 16}}{4}.$

9. $x = \dfrac{(1 \pm \sqrt{5}) \pm \sqrt{-10 \pm 2\sqrt{5}}}{4}.$

$\therefore$ the four roots are

1. $(1 + \sqrt{5}) + \sqrt{-10 + 2\sqrt{5}}.$
2. $(1 + \sqrt{5}) - \sqrt{-10 + 2\sqrt{5}}.$
3. $(1 - \sqrt{5}) + \sqrt{-10 - 2\sqrt{5}}.$
4. $(1 - \sqrt{5}) - \sqrt{-10 - 2\sqrt{5}}.$

EXAMPLES

Solve and verify:

1. $x^5 + 1 = 0.$
2. $x^2 + x + 1 = 0.$
3. $x^4 + x^3 + x^2 + x + 1 = 0.$
4. $5x^3 - 7x^2 - 7x + 5 = 0.$
5. $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 = 0.$
6. $x^5 - 1 = 0.$
7. $x^3 - x^2 + x - 1 = 0.$
8. $Ax^2 + Bx + A = 0.$
9. $x^3 - 3x^2 + 3x - 1 = 0.$
10. $4x^3 + 5x^2 - 5x - 4 = 0.$

438. A **Binomial Equation** is of the form $Ax^n \pm L = 0.$

439. PROBLEM 4. **To solve a binomial equation.**

Rule 1. *Reduce the equation to the form* $Ax^n \pm L = 0$, *and, when possible, solve by one of the methods under Problem* 3; or,

Rule 2. *Reduce the equation to the form* $x^n = \mp \dfrac{L}{A}$ *and extract the nth root of both members.*

Dem. 1. See Problem 3, Art. **437.** 2. See Dem. to Prop. 1, Art. **432.**

Model Solutions

1. Solve $x^4 = -1$.

1. $x^4 + 1 = 0.$
2. $x^4 + 2x^2 + 1 - 2x^2 = 0.$
3. $(x^2 + 1 - x\sqrt{2})(x^2 + 1 + x\sqrt{2}) = 0.$
4. $x^2 + 1 - x\sqrt{2} = 0$ and $x^2 + 1 + x\sqrt{2} = 0.$
5. $x = \dfrac{\sqrt{2} \pm \sqrt{2-4}}{2}$ and $\dfrac{-\sqrt{2} \pm \sqrt{2-4}}{2}.$
6. $x = \dfrac{\sqrt{2}(1 \pm i)}{2}$ and $\dfrac{\sqrt{2}(-1 \pm i)}{2}.$

Prove that the product of these four roots is 1.

2. Solve $x^{\frac{4}{5}} = 9$.

1. $x^{\frac{2}{5}} = \pm 3.$
2. $x^{\frac{1}{5}} = \pm\sqrt{3}$ and $\pm\sqrt{-3}.$
3. $x = \pm 9\sqrt{3}$ and $\mp 9\sqrt{-3}.$

3. Solve $x^{\frac{3}{4}} = 27$.

1. $(x^{\frac{1}{4}} - 3)(x^{\frac{2}{4}} + 3x^{\frac{1}{4}} + 9) = 0.$
2. $x^{\frac{1}{4}} = 3$ and $\dfrac{-3 \pm 3\sqrt{-3}}{2}.$
3. $x = 3^4$ and $\left(\dfrac{-3 \pm 3\sqrt{-3}}{2}\right)^4.$

4. Find the three cube roots of 8.

1. Let $x = \sqrt[3]{8}.$
2. $x^3 = 8.$
3. $x^3 - 8 = 0.$
4. $(x - 2)(x^2 + 2x + 4) = 0.$
5. $x = 2$ and $-1 \pm \sqrt{-3}.$
6. $x = 2,\ -1 + \sqrt{-3},$ and $-1 - \sqrt{-3}.$

What is the product of these three roots?

EXAMPLES

Solve for as many roots as possible by elementary methods:

1. $x^3 = 1$.
2. $x^3 = -1$.
3. $x^3 = -8$.
4. $10\,x^{\frac{2}{3}} = 90$.
5. $x^4 = 1$.
6. $2\,x^{\frac{3}{5}} = 54$.
7. $x^5 - 1 = 0$.
8. $5\,x^4 + 5 = 0$.
9. $x^5 + 1 = 0$.
10. $x^6 - 1 = 0$.
11. $x^6 + 1 = 0$.
12. $x^3 + 27 = 0$.
13. What are the three cube roots of 1?
14. What are the five fifth roots of $+1$?
15. What are the three cube roots of -1?
16. What are the four fourth roots of $+1$?
17. What are the four fourth roots of -16?

SECTION IV

SIMULTANEOUS QUADRATIC EQUATIONS

440. Prop. 1. The solution of a set of two simultaneous equations involving two unknown quantities, when one of them is a quadratic and the other a linear equation, requires the solution of a quadratic, and is, therefore, always possible.

441. PROBLEM 1. **To solve equations under Prop. 1.**

Rule. *From the linear equation find the value of one of the unknown quantities in terms of the other unknown quantity and the known quantities. Substitute this value in the quadratic equation, and solve the resulting equation.*

Dem. of Rule. Let

1. $ax + by + c = 0$ be the linear equation, and
2. $Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$ be the quadratic.

3. $y = -\dfrac{ax+c}{b}$, from (1).

4. $Ax^2 + 2Hx\left(-\dfrac{ax+c}{b}\right) + B\left(-\dfrac{ax+c}{b}\right)^2$
$$+ 2Gx + 2F\left(-\frac{ax+c}{b}\right) + C = 0.$$

5. $Ab^2x^2 - 2abHx^2 - 2bcHx + a^2Bx^2 + 2acBx + Bc^2$
$$+ 2b^2Gx - 2abFx - 2bcF + b^2C = 0.$$

6. $(Ab^2 - 2abH + a^2B)x^2 + 2(aBc - abF - bcH + b^2G)x$
$$+ (Bc^2 - 2bcF + b^2C) = 0.$$

Let l = the coefficient of x^2, m of x, and let n = the absolute term.

7. Then $lx^2 + mx + n = 0$, a quadratic equation from which

8. $x = \dfrac{-m \pm \sqrt{m^2 - 4ln}}{2l}$.

Dem. of Prop. 1. Since equation (1) is the most general form of a linear equation involving two unknown quantities, and equation (2) is the most general form of a quadratic equation involving two unknown quantities, and since their solution is seen to require the solution of a quadratic and nothing higher, the solution of such a problem is always possible.

Model Solution

Solve $\begin{cases} (1)\ \ 5x - 2y = 4, \\ (2)\ \ \frac{3}{2}x^2 - xy + \frac{2}{3}y^2 = 6. \end{cases}$

3. $x = \frac{1}{5}(2y + 4)$, from (1).
4. $9x^2 - 6xy + 4y^2 = 36$, from (2).
5. $\frac{9}{25}(4y^2 + 16y + 16) - \frac{6}{5}(2y + 4)y + 4y^2 = 36.$
6. $19y^2 + 6y - 189 = 0.$
7. $(y - 3)(19y + 63) = 0.$
8. $y = 3$ and $-3\frac{6}{19}$.

Substituting these values in (3),

9. $x = \frac{1}{5}(6 + 4)$ and $\frac{1}{5}(-\frac{126}{13} + 4)$, or,

10. $x = 2$ and $-\frac{18}{13}$.

442. Cor. In cases of fractional exponents the solution is generally made easier by putting the unknown quantities with the lowest exponents equal to the first power of new unknown quantities, and the other unknown quantities equal to corresponding powers.

Model Solution

Solve $\begin{cases} (1) \quad x^{\frac{1}{4}} + y^{\frac{1}{3}} = 5, \\ (2) \quad x^{\frac{2}{4}} + y^{\frac{2}{3}} = 13. \end{cases}$

3. Let $x^{\frac{1}{4}} = z$ and $y^{\frac{1}{3}} = w$.
4. Then $z + w = 5$ and $z^2 + w^2 = 13$.
5. $w = 5 - z$ and $z^2 + (5 - z)^2 = 13$.
6. $z^2 - 5z + 6 = 0$.
7. $z = 2$ and 3.
8. $w = 3$ and 2.
9. But $z = x^{\frac{1}{4}} = 2$ and 3. $\therefore x = 16$ and 81.
10. And $w = y^{\frac{1}{3}} = 3$ and 2. $\therefore y = 27$ and 8.

EXAMPLES

1. $\begin{cases} x + y = 7, \\ xy = 12. \end{cases}$

2. $\begin{cases} x^{\frac{2}{3}} + y = 13, \\ x^{\frac{1}{3}} + y^{\frac{1}{2}} = 5. \end{cases}$

3. $\begin{cases} x - y = 12, \\ x^2 + y^2 = 74. \end{cases}$

4. $\begin{cases} x + y = 2a, \\ x^2 + y^2 = 2a^2. \end{cases}$

5. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5, \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = 13. \end{cases}$

6. $\begin{cases} x^{\frac{1}{2}} - y^{\frac{1}{2}} = 2, \\ x + y = 34. \end{cases}$

7. $\begin{cases} 3x + 2y = 13, \\ x^2 + xy = 15. \end{cases}$

8. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{5} = 5, \\ \dfrac{2}{x} + \dfrac{5}{y} = \dfrac{5}{6}. \end{cases}$

9. $\begin{cases} x - y = 15, \\ \frac{1}{2}x = y^2. \end{cases}$

10. $\begin{cases} x + y = 3\frac{3}{4}, \\ xy = x - y. \end{cases}$

11. $\begin{cases} x + 3y = 10, \\ x^2 - 3y^2 = -27. \end{cases}$

12. $\begin{cases} 3x^2 + 3xy = 40y, \\ x - y = 2. \end{cases}$

13. $\begin{cases} x^{\frac{1}{2}} - y^{\frac{1}{3}} - 1 = 0, \\ x - y^{\frac{2}{3}} - 5 = 0. \end{cases}$

14. $\begin{cases} x^2 - 2xy - y^2 = 1, \\ x + y = 2. \end{cases}$

15. $\begin{cases} x + 4y = 2 - x, \\ 2x^2 + y^2 - xy + 5x + 3y = 34. \end{cases}$

443. Prop. 2. The solution of a set of two simultaneous equations, involving two unknown quantities, when both are quadratics, requires the solution of a biquadratic which, in general, cannot be solved by the method of quadratics.

Dem. Let 1. $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$

and 2. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

$$x = \frac{-(2hy + 2g) \pm \sqrt{(2hy + 2g)^2 - 4a(by^2 + 2fy + c)}}{2a}, \text{ from (2).}$$

Substituting this value of x in (1) gives an equation of the form $my^2 + ny = \pm(dy + e)\sqrt{ky^2 + py + q}$, which when rationalized is an equation of the fourth degree and which, in general, cannot be solved by quadratics.

Note. Only special cases, in which the system can be solved by the method of quadratics, will be given.

444. A **Homogeneous Equation** is one which has the same number of unknown factors in every term.

ILLUSTRATIONS. $x^2 + y^2 = xy$, $Ax^2 + 2Hxy + By^2 = 0$.
$x^2 + xy - y^2 - 10 = 0$ is homogeneous with the exception of the absolute term, -10.

445. Prop. 3. The solution of a set of two simultaneous equations, involving two unknown quantities, when both are quadratics and one of them is homogeneous, is always possible.

Dem. Let 1. $Ax^2 + 2Hxy + By^2 = 0$, and

2. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

3. $A\left(\frac{x^2}{y^2}\right) + 2H\left(\frac{x}{y}\right) + B = 0$, by dividing 1 by y^2.

4. $\frac{x}{y} = \frac{-2H \pm \sqrt{4H^2 - 4AB}}{2A}$, by formula.

5. $x = \left(\frac{-2H \pm \sqrt{4H^2 - 4AB}}{2A}\right)y.$

Substituting each value of x in (2) gives two equations of the form $my^2 + ny + c = 0$ and $m'y^2 + n'y + c = 0$, which are readily solved by the method of quadratics.

Rule. Let the student supply a rule for solving the problem involved in Prop. 3.

Model Solution

Solve $\begin{cases} (1)\ 2x^2 - 3xy + y^2 = 0, \\ (2)\ 3x^2 - 2xy - 3y^2 + 5x + 4y + 26 = 0. \end{cases}$

3. $(2x - y)(x - y) = 0$, from (1).

4. $x = y$ and $\frac{1}{2}y$.

5. $3y^2 - 2y^2 - 3y^2 + 5y + 4y + 26 = 0$, substituting y for x in (2).

6. $-2y^2 + 9y + 26 = 0.$

7. $2y^2 - 9y - 26 = 0.$

8. $(2y - 13)(y + 2) = 0.$

9. $y = 6\frac{1}{2}$ and $-2.$

10. $\frac{4}{9}y^2 - y^2 - 3y^2 + \frac{5}{3}y + 4y + 26 = 0$, substituting $\frac{2}{3}y$ for x in (2).

11. $$-\frac{13}{4}y^2 + \frac{13}{2}y + 26 = 0.$$

12. $$y^2 - 2y - 8 = 0.$$

13. $$(y-4)(y+2) = 0.$$

14. $$y = 4 \text{ and } -2.$$

15. If $$x = y \text{ and } \tfrac{1}{2}y,$$

16. then $$x = 6\tfrac{1}{2},\ -2 \text{ and } 2,\ -1.$$

EXAMPLES

Solve and verify:

1. $\begin{cases} 6x^2 - 5xy - 6y^2 = 0, \\ x^2 + xy + x + y = 10. \end{cases}$

2. $\begin{cases} 5x^2 - 2xy - 7y^2 = 0, \\ xy + y^2 = 60. \end{cases}$

3. $\begin{cases} 6x^2 - 5xy + y^2 = 0, \\ 2x^2 + 3xy + 5y^2 = 28. \end{cases}$

4. $\begin{cases} 25x^2 + 25xy - 24y^2 = 0, \\ x^2 + xy + 2y^2 = 74. \end{cases}$

5. $\begin{cases} 2x^2 + 3xy = 0, \\ 5xy + 7y^2 + 3x - 5y = 7. \end{cases}$

6. $\begin{cases} 6x^2 - 41xy + 35y^2 = 0, \\ xy - y^2 = 12. \end{cases}$

7. $\begin{cases} 4x^2 - 16xy - 65y^2 = 0, \\ 6x^2 + 11xy - 10y^2 = 35. \end{cases}$

8. $\begin{cases} 3x^2 - 7xy + 4y^2 = 0, \\ 9\sqrt{x^3} - 2y^2 - 9y\sqrt{x} = 0. \end{cases}$

9. $\begin{cases} x^2 - y^2 = 0, \\ 4x^2 - 3xy + y^2 + 5x - 3y = 5. \end{cases}$

10. $\begin{cases} 133x^2 + 73xy - 120y^2 = 0, \\ xy + 2y^2 = 133. \end{cases}$

446. Prop. 4. The solution of a set of two simultaneous equations, involving two unknown quantities, when both are quadratics, and both homogeneous with the exception of the absolute term, is always possible. Such solution can be accomplished by substituting for the first of the unknown quantities the product of the second and a new unknown quantity, eliminating the second, and then solving for the third, second, and first in the order named.

Dem. 1. Let

1. $$ax^2 + hxy + by^2 = c, \text{ and}$$

2. $$a'x^2 + h'xy + b'y^2 = c'.$$

3. Let $$x = vy.$$

4. Then $$av^2y^2 + hvy^2 + by^2 = c, \text{ and}$$

5. $$a'v^2y^2 + h'vy^2 + b'y^2 = c'.$$

By comparison

6. $y^2 = \dfrac{c}{av^2 + hv + b} = \dfrac{c'}{a'v^2 + h'v + b'}.$ Why?

7. $ac'v^2 + hc'v + bc' = a'cv^2 + h'cv + b'c.$

8. $(ac' - a'c)v^2 + (hc' - h'c)v + (bc' - b'c) = 0.$

9. $v = \dfrac{-(hc' - h'c) \pm \sqrt{(hc' - h'c)^2 - 4(ac' - a'c)(bc' - b'c)}}{2(ac' - a'c)}.$

Substituting these values of v in (6) gives two values of y, which in turn being substituted in $x = vy$ give two values of x. Hence the solution is always possible.

Dem. 2. Let

1. $$ax^2 + hxy + by^2 = c, \text{ and}$$

2. $$a'x^2 + h'xy + b'y^2 = c'.$$

Multiplying diagonally across the sign, =,

3. $ac'x^2 + hc'xy + bc'y^2 = a'cx^2 + h'cxy + b'cy^2.$

4. $(ac' - a'c)x^2 + (hc' - h'c)xy + (bc' - b'c)y^2 = 0.$

This equation is now homogeneous, and the example comes under Prop. 3, which has already been proved.

Rule. Let the student make a rule.

Model Solutions

1. Solve $\begin{cases} (1) \ 2x^2 - 3xy + 4y^2 = 24, \\ (2) \ 3x^2 - 5y^2 = 28. \end{cases}$

3. Let $x = vy.$

4. $2v^2y^2 - 3vy^2 + 4y^2 = 24.$

5. $3v^2y^2 - 5y^2 = 28.$

6. $y^2 = \dfrac{24}{2v^2 - 3v + 4} = \dfrac{28}{3v^2 - 5}.$

7. $\dfrac{6}{2v^2 - 3v + 4} = \dfrac{7}{3v^2 - 5}.$

8. $18v^2 - 30 = 14v^2 - 21v + 28.$

9. $4v^2 + 21v - 58 = 0.$

10. $(4v + 29)(v - 2) = 0.$

11. $v = 2$ and $-\frac{29}{4}.$

12. $y^2 = \dfrac{28}{3v^2 - 5} = \dfrac{28}{12 - 5}$ and $\dfrac{28}{\frac{2523}{16} - 5}.$

13. $y^2 = 4$ and $\frac{64}{349}.$

14. $y = \pm 2$ and $\pm \frac{8}{349}\sqrt{349}.$

15. $x = vy.$

16. $x = (2)(\pm 2) = \pm 4.$

17. Also $x = (-\frac{29}{4})(\pm \frac{8}{349}\sqrt{349}) = \mp \frac{58}{349}\sqrt{349}.$

2. Solve $\begin{cases} (1) \ x^2 + 9\frac{1}{2}xy + 2y^2 = 5, \\ (2) \ 3x^2 + 2xy + 4y^2 = 4. \end{cases}$

3. $4x^2 + 38xy + 8y^2 = 15x^2 + 10xy + 20y^2.$

4. $11x^2 - 28xy + 12y^2 = 0.$

5. $(x - 2y)(11x - 6y) = 0.$

6. $x = 2y$ and $\frac{6}{11}y.$

Substituting these values in equation (2),

7. $12y^2 + 4y^2 + 4y^2 = 4$ and $\frac{108}{121}y^2 + \frac{12}{11}y^2 + 4y^2 = 4.$

8. $20y^2 = 4$ and $\frac{724}{121}y^2 = 4.$

9. $y^2 = \frac{1}{5}$ and $\frac{121}{181}.$

10. $y = \pm\frac{1}{5}\sqrt{5}$ and $\pm\frac{11}{181}\sqrt{181}.$

11. $\begin{cases}\text{When } x = 2y,\\ \text{then } x = \pm\frac{2}{5}\sqrt{5}.\end{cases}$ 12. $\begin{cases}\text{When } x = \frac{6}{11}y,\\ \text{then } x = \pm\frac{6}{181}\sqrt{181}.\end{cases}$

Rule. Let the student make a rule.

EXAMPLES

Solve and verify:

1. $\begin{cases}x^2 + xy = a,\\ xy + y^2 = b.\end{cases}$

2. $\begin{cases}xy = 28,\\ x^2 + y^2 = 65.\end{cases}$

3. $\begin{cases}x^2 + xy = 12,\\ xy - 2y^2 = 1.\end{cases}$

4. $\begin{cases}x^2 + xy = 60,\\ xy + 2y^2 = 113.\end{cases}$

5. $\begin{cases}4x^2 + 5y^2 = 216,\\ (2x - y)^2 = 144.\end{cases}$

6. $\begin{cases}3x^2 + 8y^2 = 14,\\ x^2 + 4y^2 + xy = 6.\end{cases}$

7. $\begin{cases}2y^2 - 4xy + 3x^2 = 17,\\ x^2 - y^2 = -16.\end{cases}$

8. $\begin{cases}3x^2 + 11xy - 4y^2 = 126,\\ x^2 - 16y^2 = 9.\end{cases}$

9. $\begin{cases}2x^2 - xy = 6,\\ 2y^2 + 3xy = 8.\end{cases}$

10. $\begin{cases}6x^2 - 5xy = 21,\\ 2x^2 + 3xy = 9.\end{cases}$

11. $\begin{cases}x^2 + xy + y^2 = 52,\\ xy - x^2 = 8.\end{cases}$

12. $\begin{cases}15 - 2xy + y^2 = 0,\\ y^2 - xy + x^2 = 21.\end{cases}$

13. $\begin{cases}x^2 + xy - 6y^2 = 0,\\ 5xy + 6y^2 + x^2 = 30.\end{cases}$

14. $\begin{cases}6xy - 2x^2 - y^2 = 31,\\ x^2 - 3xy + 3y^2 = 7.\end{cases}$

15. $\begin{cases}3x^2 - 3xy + y^2 = 21,\\ 2xy = 3y^2 + x^2 - 19.\end{cases}$

16. $\begin{cases}2x^2 - xy - 15y^2 = 7,\\ 6x^2 + 11xy - 10y^2 = 35.\end{cases}$

447. Prop. 5. The solution of a set of two simultaneous equations, involving two unknown quantities, when both are symmetrical and of a degree not higher than the second, is always possible. Such solution can be accomplished by substituting for the unknown quantities the sum and the difference respectively of two other quantities.

Dem. Solve
$$\begin{cases} (1) \quad ax^2 + hxy + ay^2 + gx + gy + c = 0, \\ (2) \quad a_1x^2 + h_1xy + a_1y^2 + g_1x + g_1y + c_1 = 0. \end{cases}$$

Are these equations symmetrical with respect to x and y? Why?

Let $\qquad x = m + n$ and $y = m - n$.

$$\begin{cases} 3. \quad am^2 + 2\,amn + an^2 + hm^2 - hn^2 + am^2 - 2\,amn \\ \qquad\qquad + an^2 + gm + gn + gm - gn + c = 0, \\ 4. \quad a_1m^2 + 2\,a_1mn + a_1n^2 + h_1m^2 - h_1n^2 + a_1m^2 - 2\,a_1mn \\ \qquad\qquad + a_1n^2 + g_1m + g_1n + g_1m - g_1n + c_1 = 0. \end{cases}$$

$$\begin{cases} 5. \quad (2\,a + h)m^2 + (2\,a - h)n^2 + 2\,gm + c = 0, \\ 6. \quad (2\,a_1 + h_1)m^2 + (2\,a_1 - h_1)n^2 + 2\,g_1m + c_1 = 0. \end{cases}$$

7. $$n^2 = \frac{-(2\,a + h)m^2 - 2\,gm - c}{2\,a - h}$$
$$= \frac{-(2\,a_1 + h_1)m^2 - 2\,g_1m - c_1}{2\,a_1 - h_1}.$$

Equation (7) can be reduced to the form

8. $$Am^2 + Bm + C = 0,$$

and can be readily solved for m, by the formula.

After the values of m have been found, those of n may be found by substituting in equation (7).

Hence the solution of the equations is possible, and may be accomplished by the proposed method.

Model Solution

Solve $\begin{cases} (1) & x^3 + y^3 = 35, \\ (2) & x + y = 5. \end{cases}$

Let $x = m + n$ and $y = m - n$.

3. $m + n + m - n = 5,$
4. $m^3 + 3m^2n + 3mn^2 + n^3 + m^3 - 3m^2n + 3mn^2 - n^3 = 35.$
5. $m = \frac{5}{2},$
6. $2m^3 + 6mn^2 = 35.$
7. $\frac{125}{4} + 15n^2 = 35.$
8. $n^2 = \frac{1}{4}.$
9. $n = \pm\frac{1}{2}.$
10. $\therefore\ x = m + n = \frac{5}{2} \pm \frac{1}{2} = 3$ and 2, and
11. $y = m - n = \frac{5}{2} - (\pm\frac{1}{2}) = 2$ and 3.

EXAMPLES

Solve and verify:

1. $\begin{cases} x^3 + y^3 = 56, \\ x + y = 2. \end{cases}$

2. $\begin{cases} x + y = 5, \\ x^2 + y^2 = 17. \end{cases}$

3. $\begin{cases} \frac{1}{x} + \frac{1}{y} = a, \\ \frac{1}{x^2} + \frac{1}{y^2} = b. \end{cases}$

4. $\begin{cases} x^3 + y^3 = 504, \\ x^2 - xy + y^2 = 84. \end{cases}$

5. $\begin{cases} x + y + 3 = 0, \\ \frac{1}{x} + \frac{1}{y} - \frac{1}{6} = 0. \end{cases}$

6. $\begin{cases} x - y = 10, \\ x^2 + y^2 = 58. \end{cases}$

7. $\begin{cases} x + y = 8, \\ x^4 + y^4 = 706. \end{cases}$

8. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 7, \\ \frac{1}{x^2} + \frac{1}{y^2} = 25. \end{cases}$

9. $\begin{cases} x^3 - y^3 = 98, \\ x - y = 2. \end{cases}$

10. $\begin{cases} x + y = 12, \\ x^2 + y^2 = 74. \end{cases}$

448. Prop. 6. When the members of one equation can be divided by the members of the second equation, the system can generally be solved by making of the same degree the new equation thus obtained and one of the given equations, and then eliminating one or more terms by addition or subtraction.

MODEL SOLUTION

Solve $\begin{cases} (1)\ x^3 + y^3 = 35, \\ (2)\ x + y = 5. \end{cases}$

3. $x^2 - xy + y^2 = 7$, by dividing (1) by (2).
4. $x^2 + 2xy + y^2 = 25$, by squaring (2).
5. $3xy = 18$, by subtracting (3) from (4).
6. $xy = 6$.
7. $y = \frac{6}{x}$.
8. $x + \frac{6}{x} = 5$, by substituting (7) in (2).
9. $x^2 - 5x + 6 = 0$.
10. $x = 2$ and 3.
11. $y = \frac{6}{2} = 3$, and $\frac{6}{3} = 2$.

Why not substitute (10) in (1) instead of in (2)?

EXAMPLES

Solve and verify:

1. $\begin{cases} x - y = 2, \\ x^3 - y^3 = 8. \end{cases}$

2. $\begin{cases} x - y = 2, \\ x^3 - y^3 = 7xy. \end{cases}$

3. $\begin{cases} x^3 + y^3 = 407, \\ x^2 - xy + y^2 = 37. \end{cases}$

4. $\begin{cases} x^4 + x^2y^2 + y^4 = 364, \\ x^2 + xy + y^2 = 26. \end{cases}$

5. $\begin{cases} x + y = 5, \\ x^3 + y^3 = 65. \end{cases}$

6. $\begin{cases} x^5 + y^5 = 33, \\ x + y = 3. \end{cases}$

7. $\begin{cases} x^{\frac{1}{3}} - y^{\frac{1}{3}} = 4, \\ x - y = 208. \end{cases}$

8. $\begin{cases} x + y = 72, \\ \sqrt[3]{x} + \sqrt[3]{y} = 6. \end{cases}$

9. $\begin{cases} x^{\frac{1}{2}} + y^{\frac{1}{3}} = 5, \\ x^{\frac{3}{2}} + y = 35. \end{cases}$

10. $\begin{cases} x + y = 35, \\ \sqrt[3]{x} + \sqrt[3]{y} = 5. \end{cases}$

11. $\begin{cases} x^{\frac{1}{3}} - y^{\frac{1}{2}} + 1 = 0, \\ x^{\frac{4}{3}} - y^2 + 65 = 0. \end{cases}$

12. $\begin{cases} x^4 + x^2y^2 + y^4 = 481, \\ x^2 - xy + y^2 = 13. \end{cases}$

449. Prop. 7. When one equation of a system can be resolved into two or more linear or quadratic equations, the system can be solved by one of the preceding methods.

Model Solution

Solve $\begin{cases} (1)\ x + y + \sqrt{x+y} = 6, \\ (2)\ x^2 + y^2 = 10. \end{cases}$

3. $(\sqrt{x+y} - 2)(\sqrt{x+y} + 3) = 0.$
4. $x + y = 4$ and $9.$
5. $y = 4 - x$ and $9 - x.$
6. $x^2 + 16 - 8x + x^2 = 10$ and $x^2 + 81 - 18x + x^2 = 10.$
7. $2x^2 - 8x + 6 = 0$ and $2x^2 - 18x + 71 = 0.$
8. $x = 1$ and 3, and $\dfrac{9 \pm \sqrt{-61}}{2}.$
9. $y = 3$ and 1, and $\dfrac{9 \mp \sqrt{-61}}{2}.$

EXAMPLES

Solve and verify:

1. $\begin{cases} \dfrac{x^2}{y^2} + \dfrac{4x}{y} - \dfrac{85}{9} = 0, \\ x - y = 2. \end{cases}$

2. $\begin{cases} 4xy + x^2y^2 - 96 = 0, \\ x + y = 6. \end{cases}$

3. $\begin{cases} x^2y^4 - 7xy^2 - 945 = 765, \\ xy - y = 12. \end{cases}$

4. $\begin{cases} x + y + \sqrt{x+y} - 12 = 0, \\ xy = 20. \end{cases}$

5. $\begin{cases} \dfrac{9x}{y} = \dfrac{4y}{x}, \\ y + 2x + 3xy = 485. \end{cases}$

6. $\begin{cases} 3x^2 - 7xy + 4y^2 = 0, \\ 9x^{\frac{3}{2}} - 9y\sqrt{x} - 2y^2 = 0. \end{cases}$

7. $\begin{cases} x^2 + 2xy + y^2 + 2x + 2y = 120, \\ xy - y^2 = 8. \end{cases}$

8. $\begin{cases} x - 2\sqrt{xy} + y - \sqrt{x} + \sqrt{y} = 0, \\ \sqrt{x} + \sqrt{y} = 5. \end{cases}$

450. Prop. 8. When the addition, or subtraction, of the corresponding members of two simultaneous equations, or of any multiple of them, produces an equation which can be factored, the system can generally be solved according to Proposition 7 by comparing the new equations obtained by factoring, with one of the given equations.

Model Solution

$$\text{Solve } \begin{cases} (1)\ x^2+x+y+y^2=18, \\ (2)\ xy=6. \end{cases}$$

3. $x^2+y^2+x+y=18$
4. $2xy \qquad\qquad =12$
5. $x^2+2xy+y^2+x+y=30.$
6. $(x+y)^2+(x+y)-30=0.$
7. $(x+y-5)(x+y+6)=0.$
8. $x+y=5$ and $x+y=-6.$
9. $y=5-x$ and $y=-6-x.$
10. $5x-x^2=6$ and $-6x-x^2=6$, substituting (9) in (2).
11. $x^2-5x+6=0$ and $x^2+6x+6=0.$
12. $(x-2)(x-3)=0$ and $x=\dfrac{-6\pm\sqrt{36-24}}{2}.$
13. $x=2,\ 3$ and $-3\pm\sqrt{3}.$
14. $y=5-2=3$ and $y=-6-(-3\pm\sqrt{3}).$
15. $y=5-3=2$ and $y=-3\mp\sqrt{3}.$

EXAMPLES

Solve and verify:

1. $\begin{cases} y^2+x-3y=0, \\ y+x-x^2=0. \end{cases}$

2. $\begin{cases} x^{\frac{4}{3}}+y^{\frac{2}{3}}+x^{\frac{2}{3}}+y^{\frac{1}{3}}=26, \\ x^{\frac{2}{3}}y^{\frac{1}{3}}=8. \end{cases}$

3. $\begin{cases} x^2+y^2+3x+3y=64, \\ xy=12. \end{cases}$

4. $\begin{cases} x^3+y^3=189, \\ x^2y+y^2x=180. \end{cases}$

5. $\begin{cases} x^2+y^2-x-y=78, \\ xy+x+y=39. \end{cases}$

6. $\begin{cases} x^2+4y^2+6y-3x=292, \\ xy=51. \end{cases}$

451. Prop. 9. When one part of an equation containing the unknown quantities is the reciprocal of the remaining part, the system can generally be solved by representing one part by a new unknown quantity, and the other part by the reciprocal of such quantity. The solutions obtained for this new equation, compared with the remaining given equation, will be the required solutions.

MODEL SOLUTION

$$\text{Solve}\begin{cases}(1)\quad \dfrac{x+y}{x-y}+10\cdot\dfrac{x-y}{x+y}=7,\\ (2)\quad x^2-y^2=8.\end{cases}$$

3. Let $\dfrac{x+y}{x-y}=z.$

4. Then $z+\dfrac{10}{z}=7.$

5. $z^2-7z+10=0.$

6. $z=2$ and $5.$

7. But $z=\dfrac{x+y}{x-y}=2$ and $5.$

8. $\therefore\ x=3y$ and $2x=3y.$

9. $(3y)^2-y^2=8$ and $\left(\dfrac{3y}{2}\right)^2-y^2=8.$

10. $y=\pm 1$ and $\pm\frac{4}{5}\sqrt{10}.$

11. $x=\pm 3$ and $\pm\frac{6}{5}\sqrt{10}.$

EXAMPLES

Solve and verify:

1. $\begin{cases}\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{5}{2},\\ x+y=10.\end{cases}$

2. $\begin{cases}x^2y-xy^2=96,\\ \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{50}{7}.\end{cases}$

3. $\begin{cases}\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{89}{40},\\ 6x-20y=9.\end{cases}$

4. $\begin{cases}\left(\dfrac{3x-3y}{x+y}\right)^2+\left(\dfrac{3x+3y}{x-y}\right)^2=82,\\ xy=2.\end{cases}$

452. PROBLEM 2. **To extract the square root of a binomial surd by means of two simultaneous quadratic equations.**

Dem. See Art. **413**.

MODEL SOLUTION

Find the square root of $31 + 42\sqrt{-2}$.

1. Let $\sqrt{x} + \sqrt{y} = \sqrt{31 + 42\sqrt{-2}}$.
2. $x + y + 2\sqrt{xy} = 31 + 42\sqrt{-2}$.
3. $x + y = 31$ and $2\sqrt{xy} = 42\sqrt{-2}$.
4. $y = 31 - x, \quad xy = -882$.
5. $x(31 - x) = -882$.
6. $x^2 - 31x - 882 = 0$.
7. $(x - 49)(x + 18) = 0$.
8. $x = 49$ and -18.
9. $y = -18$ and 49.
10. Hence $\sqrt{x} + \sqrt{y} = \sqrt{49} + \sqrt{-18}$, or $\sqrt{-18} + \sqrt{49}$
11. $= \pm 7 \pm 3\sqrt{-2}$, or $\pm 3\sqrt{-2} \pm 7$
12. $= \pm(7 + 3\sqrt{-2})$, or $\pm(3\sqrt{-2} + 7)$.

EXAMPLES

Find the square root and prove each:

1. $8\sqrt{-1}$.
2. $2 + \frac{1}{2}\sqrt{18}$.
3. $0 - \sqrt{-1}$.
4. $-3 + \sqrt{-16}$.
5. $-2 - 2\sqrt{-15}$.
6. $0 + \sqrt{-1}$.
7. $1 + \sqrt{1 - z^2}$.
8. $2 + 4\sqrt{-42}$.
9. $30\frac{1}{2} - 10\sqrt{\frac{6}{10}}$.
10. $-2 + 4\sqrt{-6}$.
11. $\frac{9}{8} - \sqrt{\frac{9}{8}}$.
12. $2cd\sqrt{-1}$.
13. $5\sqrt{2} - 2\sqrt{12}$.
14. $22 + 10\sqrt{-3}$.
15. $20 - 10\sqrt{-5}$.

MISCELLANEOUS EXAMPLES

Solve and verify:

1. $\begin{cases} x - y = 3, \\ x^5 - y^5 = 3093. \end{cases}$

2. $\begin{cases} x - y = 8, \\ x^4 - y^4 = 1456. \end{cases}$

3. $\begin{cases} y^2 - 64 = 8\,x^{\frac{1}{2}}y, \\ y - 4 = 2\,x^{\frac{1}{2}}y^{\frac{1}{2}}. \end{cases}$

4. $\begin{cases} x^2y^4 + y^2 = 333, \\ xy^2 + y = 21. \end{cases}$

5. $\begin{cases} x - y + 18 = 0, \\ 10\,x + y = 3\,xy. \end{cases}$

6. $\begin{cases} x^2(x - y) = 75, \\ x^2(2\,x + 3\,y) = 400. \end{cases}$

7. $\begin{cases} x^2 - xy + y^2 = 7, \\ x^4 + x^2y^2 + y^4 = 133. \end{cases}$

8. $\begin{cases} \dfrac{x}{y} - \dfrac{y}{x} = \frac{9}{20}, \\ 2\sqrt{x^2 - y^2} + xy = 26. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 41, \\ x + y = 12 - \sqrt{x + y}. \end{cases}$

10. $\begin{cases} x + y + \sqrt{x + y} = 12, \\ x^3 + y^3 = 189. \end{cases}$

11. $\begin{cases} \sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = \dfrac{7}{\sqrt{xy}} + 1, \\ \sqrt{x^3y} + \sqrt{xy^3} = 78. \end{cases}$

12. $\begin{cases} \sqrt{\dfrac{3\,x}{x + y}} + \sqrt{\dfrac{x + y}{3\,x}} = 2, \\ xy - x - y - 54 = 0. \end{cases}$

13. $\begin{cases} xy = 2\,y^3, \\ x - y = 15. \end{cases}$

14. $\begin{cases} xy + \dfrac{x}{y} = \dfrac{5}{3}, \\ \dfrac{1}{xy} + \dfrac{y}{x} = \dfrac{20}{3}. \end{cases}$

15. $\begin{cases} x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3\,x, \\ \sqrt{x} + \sqrt{y} = x. \end{cases}$

16. $\begin{cases} x + y = 5, \\ x^2 - xy + y^2 = 7. \end{cases}$

17. $\begin{cases} x^2 + xy + y^2 = 7, \\ 5\,x + 2\,y = 9. \end{cases}$

18. $\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = \sqrt{5}, \\ x + y = 2 + \sqrt{5}. \end{cases}$

19. $\begin{cases} x + y = 20, \\ \sqrt{x} - \sqrt{y} = 2\sqrt{xy}. \end{cases}$

20. $\begin{cases} x + 4\,y = 14, \\ y^2 + 4\,x = 2\,y + 11. \end{cases}$

21. $\begin{cases} x - \sqrt{x} - 3 + y = 0, \\ y - \sqrt{y} + x - 4 = 0. \end{cases}$

22. $\begin{cases} x + y^2 = 25, \\ 13(x - y) = 5(x + y). \end{cases}$

23. $\begin{cases} (x^2 + 1)\,y = xy + 126, \\ (x^2 + 1)\,y = x^2y^2 - 744. \end{cases}$

24. $\begin{cases} x^2 - xy + y^2 = 49 - 2\,xy, \\ x^4 + x^2y^2 + y^4 = 931. \end{cases}$

25. $\begin{cases} \dfrac{ab}{xy} = 1, \\ \dfrac{a^2}{x^2} + \dfrac{y^2}{b^2} = 6. \end{cases}$

26. $\begin{cases} 3y^2 - 2x^2 = 19, \\ y^2 + xy = 15. \end{cases}$

27. $\begin{cases} 5(3x - y) = 9(x + y), \\ 3x^2 + 2xy - y^2 = 180. \end{cases}$

28. $\begin{cases} 6x^2 - 17xy + 12y^2 = 0, \\ x^2y^2 - 18xy = -72. \end{cases}$

29. $\begin{cases} x^2 + y^2 = 41, \\ x + y = \dfrac{6}{x - y} + \sqrt{\dfrac{x - y}{x + y}}. \end{cases}$

30. $\begin{cases} xy = 576, \\ \sqrt{x} + y^{\frac{1}{2}} = 14. \end{cases}$

31. $\begin{cases} 3x^2 - 4xy = 7, \\ 3xy - 4y^2 = 5. \end{cases}$

32. $\begin{cases} x + y = 10, \\ \dfrac{x^4}{y^2} + \dfrac{y^4}{x^2} + 2xy = 136\frac{1}{9}. \end{cases}$

33. $\begin{cases} xy = 12, \\ x^2y + x^2 + y^2(x + 1) = 109. \end{cases}$

34. $\begin{cases} \dfrac{x + y}{x - y} - \dfrac{x - y}{x + y} = \dfrac{24}{5}, \\ \dfrac{1}{x^2}\sqrt{x - y} + \dfrac{1}{x} = \dfrac{4}{9\sqrt{x - y}}. \end{cases}$

35. $\begin{cases} (x^2 + y^2) : (x^2 - y^2) :: 29 : 21, \\ x - 7 + y = 0. \end{cases}$

36. $\begin{cases} x^2 = 18 + 4xy - 9x, \\ \dfrac{\sqrt{3x - 2y}}{\sqrt{2x}} + \dfrac{\sqrt{2x}}{\sqrt{3x - 2y}} = 2. \end{cases}$

37. $\begin{cases} \dfrac{2x + \sqrt{y}}{2x - \sqrt{y}} = \dfrac{16}{15} + \dfrac{2x - \sqrt{y}}{2x + \sqrt{y}}, \\ 2x + y = 26 - 7\sqrt{2x + y + 4}. \end{cases}$

38. $\begin{cases} \dfrac{x^2 + y^2}{xy} = \dfrac{34}{15}, \\ \sqrt{x + y} + 2\sqrt{x - y} = \dfrac{2(x - 1)}{\sqrt{x - y}}. \end{cases}$

39. $\begin{cases} \sqrt{x} + \sqrt{y} = 6, \\ x + 4\sqrt{x} + 4y = 21 + 8\sqrt{y} + 4\sqrt{xy}. \end{cases}$

40. $\begin{cases} x^2 + y^2 - 4y = 1, \\ x^2 + 4y + \sqrt{2x^2 + 6y + 10} = y + 19. \end{cases}$

PROBLEMS INVOLVING QUADRATIC EQUATIONS

1. The sum of two numbers is 14, and the sum of their cubes is 1358. Find the numbers.

2. The sum of three consecutive numbers is 365. Find the numbers.

3. The hypotenuse of a right triangle is 10; its area is 48. Find the sides.

4. A farmer bought some sheep for $72, and found that if he had received 6 more for the same money he would have paid $1 less for each. How many did he buy?

5. A, B, and C have each the same amount of money, and spend it for oranges. A pays 6 cents more a dozen than B, and gets 3 dozen fewer. C pays 6 cents less a dozen than B, and gets 5 dozen more than B. How much money had each?

6. A broker sells some railroad shares for $3240. A few days later, the price having fallen $9 per share, he buys, for the same sum, 5 more shares than he had sold. What was the price and the number of shares transferred each day?

7. A man had a rectangular lot containing 1200 square yards. He added 3 yards to one side and took $1\frac{1}{2}$ yards from the other, thereby increasing the area of his lot 60 square yards. Find the original dimensions of the lot.

8. An army of 46,800 men consists of several brigades, each containing the same number of men. If there were two brigades more, there would be 1950 men fewer in each brigade. How many brigades are there?

9. Two boys, Joe and Harry, received a prize-box of money containing $210, from which each was to draw daily a certain fixed sum. The boys emptied the box in six weeks. Find the sum which each boy drew daily from the box, if Joe drawing alone would have emptied it five weeks earlier than Harry drawing alone.

10. From a sheet of paper 14 inches long a border of uniform width was cut all around, thereby reducing the area $\frac{5}{8}$. Had the sheet been 3 inches narrower, and had a border of the same width been cut away, the area would have been reduced $\frac{4}{7}$. What was the width of the sheet?

11. A distributes \$180 in equal sums among a certain number of people. B distributes the same sum, but gives to each person \$6 more, and to 40 persons fewer than A. Find what A gives to each person.

12. A boat's crew row 3 miles down a river and back again in 1 hour and 4 minutes. When rowing at half this rate they go over the same course in 2 hours and 40 minutes. Find the rate of the current and the first rate of rowing.

13. A certain number composed of two digits is equal to twice the sum of its digits. The number obtained by interchanging the digits is equal to the square of the sum of the digits. Find the original number.

14. A, B, and C are three stations on a railroad. From A to B is 216 miles, from A to C is 240 miles. Two trains, Nos. 5 and 6, on parallel tracks, pass A simultaneously at 12 o'clock noon. The rate of train 6 is 8 miles an hour less than that of train 5; and train 6 passes B $1\frac{1}{2}$ hours later than No. 5 passes C. Find the rate and direction of each train; the hour at which train 5 passes C; and the hour at which train 6 passes B.

15. A man walks at a regular rate of speed on a road which passes over a bridge, distant 21 miles from the point which he has reached at noon. If his rate of speed were half a mile an hour greater than it is, the time at which he crosses the bridge would be an hour earlier. Find his actual rate of speed, and the time at which he crosses the bridge.

16. The fore wheel of a carriage turns 132 times more than the hind wheel in going a mile. But if each were increased 2 feet in circumference, the fore wheel would turn but 88 times more. Find the circumference of each.

17. The sum of two numbers is 7, and the sum of their 4th powers is 641. What are the numbers?

18. The sum of two numbers is 6, and the sum of their 5th powers is 1056. What are the numbers?

19. A and B have each a square field. The combined area is 1300 square rods, and 200 rods of fence are required to inclose both. What is the value of each, at $2.25 per square rod?

20. A and B engaged to work for a certain number of days. A lost 4 days of the time and received $18.75. B lost 7 days and received $12. Now had A lost 7 and B 4 days, the amounts received would have been equal. How long did they engage to work and at what rates?

21. From a vat filled with 256 gallons of wine a quantity of wine is drawn, and the vat is refilled with water. The same quantity is again drawn, and the vat is refilled with water. After four drafts there are left 81 gallons of wine. How much wine is drawn each time?

22. A and B, two workmen of unequal skill, finish a piece of work. A receives $27, and B, who has worked 3 days fewer, receives $18.75. If B had worked for the whole time, and A 3 days less than the whole time, they would have received equal amounts. Find the number of days each has worked, and his daily wages.

23. Two travelers, A and B, set out to meet each other, A leaving the town C at the same time that B left D. They traveled the direct road from C to D, and on meeting it appeared that A had traveled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What is the distance between C and D?

24. On a trip, each member of a golf team spent the same amount of money. If there had been 5 more players on the team, and each player had spent 25 cents more, the bill would have been $33. If there had been 2 fewer players, and each player had spent 30 cents less, the bill would have been $11. Find the number of players, and what each spent.

CHAPTER XVIII

THE PROGRESSIONS

453. A **Series** is a succession of numbers that follow some fixed law; as 1, 2, 4, 8, 16, ⋯.

454. The **Terms** of a series are the successive numbers.

455. A **Finite Series** is one in which the number of terms is finite.

What is an *infinite series?* An ascending series? A descending series? Series are of many kinds, two of which will be touched upon in this treatise, *i.e.*, Arithmetical Progression and Geometrical Progression.

SECTION I

ARITHMETICAL PROGRESSION

456. An **Arithmetical Progression (A. P.)** is a series in which each term after the first is derived from the next preceding term by adding to it a constant quantity.

ILLUSTRATION. 1, 3, 5, 7, 9, ⋯.

Questions. Why is this a series? Why is it a progression? How may the second number be obtained from the first? What is the fixed law in this series? What kind of series is 7, 4, 1, -2, -5? What constant is added to each term to produce the next following term? What is the number added in $\frac{1}{5}$, $\frac{7}{10}$, $\frac{6}{5}$, $\frac{17}{10}$? Which is the 1st term? The 2d? The last? What is the sum of all the terms? Which are the means? What is the difference between any two consecutive terms? Why may this difference be called a common difference? Define common difference.

Abbreviations

f stands for the first term of the series.
l stands for the last term of the series.
n stands for the number of terms in the series.
d stands for the common difference.
s stands for the sum of the terms of the series.

Fundamental Formulas

457. The **Relations** between the first term, last term, number of terms, common difference, and the sum of all the terms may be expressed by means of two formulas:

$$\textbf{1. } \boldsymbol{l = f + (n-1)d.}$$

$$\textbf{2. } \boldsymbol{s = \tfrac{1}{2}(f + l)n.}$$

Note. Since these formulas are two equations, three of the five symbols must be known, and two may be unknown. Why? These formulas may, therefore, be treated as two equations of two unknown quantities, and solved by the four transformations.

Dem. Formula No. 1.

The general series is: f, $(f+d)$, $(f+2d)$, $(f+3d)$, ...

1. 1st term $= f$.
2. 2d term $= f + d$.
3. 3d term $= f + 2d$.
4. 4th term $= f + (4-1)d$.
5. 10th term $= f + 9d$.
6. nth term $= f + (n-1)d =$ last term $= l$.

Dem. Formula No. 2.

1. $s = f + (f+d) + (f+2d) + (f+3d) + \cdots + (l-2d) + (l-d) + l.$
2. $s = l + (l-d) + (l-2d) + (l-3d) + \cdots + (f+2d) + (f+d) + f.$
3. $2s = (f+l) + (f+l) + (f+l) + (f+l) + \cdots$
   $+ (f+l) + (f+l) + (f+l)$, by adding (1) and (2).
4. $2s = (f+l)n.$
5. $s = \frac{1}{2}(f+l)n.$

Model Solutions

1. Find l and s in 4, $5\frac{1}{4}$, $6\frac{1}{2}$, $7\frac{3}{4}$, $\cdots$ to 10 terms.

Formula 1. $l = f + (n-1)d.$

$l = 4 + (10-1)1\frac{1}{4} = 15\frac{1}{4}.$

Formula 2. $s = \frac{1}{2}(f + l)n.$

$s = \frac{1}{2}(4 + 15\frac{1}{4})10 = 96\frac{1}{4}.$

2. Given $s = 1127$, $l = 41$, $d = \frac{3}{4}$; find f and n.

$$l = f + (n-1)d.$$

1. $41 = f + (n-1)\frac{3}{4}.$

$$s = \left(\frac{f+l}{2}\right)n.$$

2. $1127 = \frac{1}{2}(f + 41)n.$

3. $164 = 4f + 3n - 3,$ from (1).

4. $2254 = fn + 41n,$ from (2).

5. $4f + 3n = 167,$ from (3).

6. $fn + 41n = 2254,$ from (4).

7. $n = \dfrac{167 - 4f}{3} = \dfrac{2254}{f + 41},$ from (5) and (6).

8. $-4f^2 + 3f + 6847 = 6762.$

9. $4f^2 - 3f - 85 = 0.$

10. $(f-5)(4f+17) = 0.$

11. $f = 5$ and $-\frac{17}{4}.$

12. $n = \dfrac{167 - 20}{3} = 49,$ from (7) and (11).

13. Also $n = \dfrac{167 + 17}{3} = \dfrac{184}{3} = 61\frac{1}{3},$ from (7) and (11).

Series 1. 5, $5\frac{3}{4}$, $6\frac{1}{2}$, $\cdots$, 41.

Proof

$s = \frac{1}{2}(5 + 41)49 = 23 \times 49 = 1127.$

Series 2. $-\frac{17}{4}, -\frac{14}{4}, -\frac{11}{4}, -\frac{8}{4}, \cdots, 41.$

Proof

$$s = \tfrac{1}{2}(-\tfrac{17}{4} + 41)61\tfrac{1}{3} = \tfrac{147}{8} \times \tfrac{184}{3} = 49 \times 23 = 1127.$$

What is there peculiar about Series 2? How should $n = 61\frac{1}{3}$ be interpreted?

3. Insert 3 arithmetical means between 4 and 20.

Formula. $l = f + (n-1)d.$

$$20 = 4 + (5-1)d. \quad n = 5. \quad \text{Why?}$$

$$4d = 16.$$

$$d = 4.$$

Series. 4, 8, 12, 16, 20.

EXAMPLES

1. $f = 1,\ d = 2,\ n = 20$; find l.
2. $f = 8,\ l = 203,\ n = 40$; find d.
3. $f = 8,\ l = 203,\ d = 5$; find n.
4. $f = 2\frac{1}{2},\ d = 1\frac{1}{2},\ n = 11$; find l.
5. $f = 2,\ l = 40,\ n = 20$; find s.
6. $f = 16,\ n = 60,\ d = 32$; find l and s.
7. In $1, \frac{1}{2}, 0, -\frac{1}{2}, \cdots$ to 12 terms, find l and s.
8. In $\frac{42}{36}, \frac{48}{36}, \frac{9}{6}, \cdots$, which term is 18?
9. In $2 + 7 + 12 + \cdots = 245$, how many terms are there?
10. The 5th term is 14, the 10th, 29. Find the first three terms.
11. $s = -\frac{55}{4},\ n = 20,\ f = \frac{1}{2}$; find d.
12. $s = 1127,\ n = 49,\ d = \frac{3}{4}$; find l and f.
13. Insert 5 arithmetical means between 3 and 21.
14. The sum of three numbers in A. P. is 24, and their product is 312. What are the numbers when represented by $x - y$, x, $x + y$?

15. The sum of four numbers in A. P. is 10, and the sum of their cubes is 100. What are the numbers if represented by $x-3y$, $x-y$, $x+y$, and $x+3y$? Why is the notation suggested in this example different from the notation suggested in the last preceding example?

16. A starts from a certain point and travels 1 mile the first hour, 2 the second, 3 the third, and so on. Three hours later B starts from the same point and follows A, traveling 10 miles an hour. In how many hours will B overtake A? In how many hours thereafter will A overtake B?

SECTION II

GEOMETRICAL PROGRESSION

458. A Geometrical Progression (G. P.) is a series in which each term after the first is derived from the next preceding term by multiplying it by a constant quantity. This constant quantity is called the *ratio*, the abbreviation for which is r.

ILLUSTRATION. 1, 2, 4, 8, 16, 32.

QUESTIONS. $f=?$; $l=?$; $r=?$; $s=?$ Why is this a series? What is the fixed law? Define the ratio. $-27, +9, -3, +1$ is what kind of series? What is its ratio?

FUNDAMENTAL FORMULAS

1. $l=fr^{n-1}$.

2. $s=\dfrac{fr^n-f}{r-1}$, or $\dfrac{lr-f}{r-1}$.

Dem. Formula No. 1.

1. 1st term $=f$.
2. 2d term $=fr$.
3. 3d term $=fr^2$.
4. 4th term $=fr^{4-1}$.
5. 10th term $=fr^{10-1}$.
6. nth or l term $=fr^{n-1}$.

Dem. Formula No. 2.

1. $s = f + fr + fr^2 + \cdots + fr^{n-2} + fr^{n-1}$.
2. $sr = \quad fr + fr^2 + \cdots + fr^{n-2} + fr^{n-1} + fr^n$. (1) $\times r$.
3. $sr - s = fr^n - f$, by subtracting (1) from (2).
4. $s = \dfrac{fr^n - f}{r-1}$; or
5. $s = \dfrac{lr - f}{r-1}$, $\because\ fr^n = fr^{n-1} \cdot r = lr$.

3. Formula for an infinite decreasing series,

$$s = \frac{f}{1-r}.$$

Dem. $l = 0$. $lr = 0$. $\therefore\ s = \dfrac{lr - f}{r-1} = \dfrac{0-f}{r-1} = \dfrac{f}{1-r}$.

Model Solutions

1. $f = 2$, $r = 3$, $n = 5$; find l.

Formula: $l = fr^{n-1}$.

$$l = 2 \cdot 3^{5-1} = 2 \cdot 81 = 162.$$

2. $f = 2$, $r = 3$, $l = 162$; find n.

Formula: $l = fr^{n-1}$.

$$162 = 2 \cdot 3^{n-1}.$$

$$3^{n-1} = 81.$$

$n - 1$ must equal 4.

$$\therefore\ n = 5.$$

3. $f = 2$, $r = 3$, $n = 5$; find s.

Formula: $s = \dfrac{fr^n - f}{r-1}$.

$$s = \frac{2 \cdot 3^5 - 2}{3-1} = 3^5 - 1 = 242.$$

4. Find the sum of the infinite series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ….

$$\text{Formula: } s = \frac{f}{1-r}.$$

$$s = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

5. Find the value of .2272727 ….

Supposed series $= .2 + .027 + .00027 + .0000027 \cdots$.

Progression $= .027 + .00027 + .0000027 \cdots$.

$$\text{Formula: } s = \frac{f}{1-r}.$$

$$s = \frac{.027}{1-.01} = \frac{27}{1000-10} = \frac{27}{990} = \frac{3}{110}.$$

$$\therefore\ .2272727 \cdots = \frac{2}{10} + \frac{3}{110} = \frac{25}{110} = \frac{5}{22}.$$

Why does not .2 belong to the progression? What is the ratio? How is it found? Why is .027 the first term?

6. Find the 6th and 11th terms of 5, 10, 20, ….

1. Call the 6th term the nth, or last term. $\therefore\ l = 5 \cdot 2^5 = 160$.
2. Call the 11th term the nth, or last term. $\therefore\ l = 5 \cdot 2^{10} = 5120$.

EXAMPLES

1. 2, $2\frac{2}{3}$, $3\frac{1}{9}$, 4. Find s and r.
2. $-\frac{1}{3}$, $\frac{1}{2}$, $-\frac{3}{4}$, …. Find r and the 10th term.
3. $\frac{2}{3}$, $\frac{1}{6}$, $\frac{1}{24}$, …. Find r and s.
4. .2333 …. Find the value of the series.
5. 128 … $\frac{1}{2}$. Insert 3 geometrical means.
6. If the arithmetical mean between x and y is double the geometrical, find $x \div y$.
7. Find the value of $3 + 1 + \frac{1}{3} + \cdots$ to infinity.

8. Find the value of $9-6+4-\cdots$ to infinity.

9. Find the value of $16-2+\frac{1}{4}-\cdots$ to infinity.

10. Find the value of $1-x+x^2-x^3+\cdots$ to infinity, if $x<1$.

11. Find the value of $2+\dfrac{2}{\sqrt{3}}+\cdots$ to infinity.

12. Find the 9th term of $2+\dfrac{2}{\sqrt{3}}+\cdots$ to infinity.

13. Find two geometrical means between 7 and 189.

14. Find two geometrical means between x and y.

15. Find the value of $15.120352035\cdots$.

16. Find the value of $53.\dot{2}7\dot{1}$.

17. The sum of three terms in G. P. is 13, and the sum of their squares is 91. What are the terms? Do x, xy, xy^2 represent them? Why? What is the ratio?

18. The sum of four terms of a G. P. is 15, and the sum of the means is 6. What are the terms? Do $\dfrac{x^2}{y}$, x, y, $\dfrac{y^2}{x}$ represent them? What is the ratio?

19. The sum of three terms in G. P. is 217, and the sum of the extremes is to the means as 26 is to 5. What are the three terms?

20. There are three numbers in geometrical progression, the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

21. There are three numbers in geometrical progression, whose continued product is 64, and the sum of their cubes is 584. Required the numbers.

22. There are three numbers in geometrical progression, whose product is 64, and sum 14. What are the numbers?

23. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers.

24. It is required to find four numbers in geometrical progression, such that their sum shall be 15, and the sum of their squares 85.

CHAPTER XIX

LOGARITHMS

459. A **Logarithm (log)** is the exponent with which a fixed number is affected in order to produce any required number.

460. A **System** of logarithms is a scheme by which all numbers may be represented to any required degree of accuracy by exponents with which a fixed number may be affected.

461. The **Base** of a system of logarithms is the fixed number which is affected with the different exponents.

462. The **Common System** of logarithms has 10 for its base. Any number greater than 1 may be used as a base, as 2.71828+, which is the base of the Napierian system.

ILLUSTRATIONS. If 10 is the base, the logarithm of 100 is 2. In other words, 10 must be affected with the exponent 2 to produce 100. What is the logarithm of 1000? With what exponent must 10 be affected to produce 1000?

If $10^x = 1000$, $x = ?$ If $10^x = 10{,}000$, $x = ?$ What is log 10,000?

Again, suppose the base of a system is 2. The logarithm of 8 is 3. Why? $\because\ 2^3 = 8$. $2^x = 16$, $x = ?$ $2^x = 32$, $x = ?$ $\log 32 = ?$ $\log 64 = ?$ $\log 128 = ?$ Why?

Suppose the base is 3. $\log 9 = ?$ $\log 81 = ?$ $\log 243 = ?$

Could the base of a system be 0, 1, or a negative quantity? Why?

$1^2 = ?$ $1^3 = ?$ $1^{100} = ?$ $0^1 = ?$ $0^5 = ?$ $(-2)^3 = ?$

If the base is -2, will $(-2)^x$ ever give 8? Would there be any logarithm of 8? Why not?

If the base is 27, what is log 3? $27^x=3$, $x=?$ $27^x=9$, $x=?$ $\log \frac{1}{3} = ?$ $\log \frac{1}{3} = x$, $27^x = \frac{1}{3}$, $27^{-\frac{1}{3}} = \frac{1}{3}$, $\therefore\ x = -\frac{1}{3} = -.33333+$. Therefore the logarithm of $\frac{1}{3}$ to base 27 is $-.33333+$.

463. The **Relations** between the number n, its logarithm x, and the base a of the logarithm are expressed by the equations $a^x = n$ and $\log_a n = x$. The second is read "the logarithm of n to the base a is x." Read $\log_2 64 = 6$ and $2^6 = 64$.

464. The **Object** of logarithms is to facilitate the processes of multiplication, division, involution, and evolution.

465. Prop. 1. The logarithm of 1 is zero.

Dem.
1. $a^0 = 1$.
2. $\therefore \log_a 1 = 0$.

ILLUSTRATION. $10^0 = 1$, $\therefore \log_{10} 1 = 0$.

466. Prop. 2. The logarithm of a base is 1.

Dem.
1. $a^1 = a$.
2. $\therefore \log_a a = 1$.

ILLUSTRATION. $10^1 = 10$, $\therefore \log_{10} 10 = 1$.

467. Prop. 3. The logarithm of a product is equal to the sum of the logarithms of its factors.

Dem. Let mn be the given product, and m and n the factors of the product; then $\log_a mn = \log_a m + \log_a n$.

1. For let $x = \log_a m$ and $y = \log_a n$;
2. then $m = a^x$ and $n = a^y$;
3. and $mn = a^x \cdot a^y = a^{x+y}$.
4. $\therefore \log_a mn = x + y = \log_a m + \log_a n$.

468. Prop. 4. The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

Dem. Let $\frac{m}{n}$ be the given quotient, m being the dividend and $n =$ the divisor; then $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$.

1. For let $x = \log_a m$ and $y = \log_a n$;
2. then $m = a^x$ and $n = a^y$;
3. and $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$.
4. $\therefore \log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n$.

469. Prop. 5. The logarithm of the nth power of any number is equal to n times the logarithm of the number.

Dem. Let m^n be the given power; then $\log_a m^n = n \log_a m$.

1. For let $m = a^x$;
2. then $m^n = a^{xn}$. Why?
3. $\therefore \log_a m^n = xn = n \log_a m$. Why?

470. Prop. 6. The logarithm of the nth root of a number is equal to $\frac{1}{n}$th of the logarithm of the number.

Dem. Let $\sqrt[n]{m}$ be the given root;

then $$\log_a \sqrt[n]{m} = \frac{1}{n} \log_a m.$$

1. For let $m = a^x$;
2. then $m^{\frac{1}{n}} = a^{\frac{x}{n}}$. Why?
3. $\therefore \log_a \sqrt[n]{m} = \frac{x}{n} = \frac{1}{n} \log_a m$. Why?

Model Solutions

Given $\log 2 = 0.30103$ and $\log 3 = 0.47712$, find the logarithms of (1) 5; (2) 225; (3) $0.003\frac{1}{3}$, to base 10.

(1) $$5 = \frac{10}{2}.$$

$$\log 5 = \log 10 - \log 2. \quad \text{Why?}$$
$$= 1 - 0.30103. \quad \text{Why?}$$
$$= 0.69897.$$

(2)
$$225 = 3^2 \times 5^2.$$
$$\log 225 = 2 \log 3 + 2 \log 5. \quad \text{Why?}$$
$$= 2\,(0.47712) + 2\,(0.69897)$$
$$= 0.95424 + 1.39794$$
$$= 2.35218.$$

(3)
$$0.003\tfrac{1}{3} = \tfrac{3\frac{1}{3}}{1000} = \tfrac{1}{300} = 3^{-1} \times 10^{-2}. \quad \text{Why?}$$
$$\log 0.003\tfrac{1}{3} = \log (3^{-1} \times 10^{-2})$$
$$= - \log 3 - 2 \log 10. \quad \text{Why?}$$
$$= - 0.47712 - 2$$
$$= - 2.47712.$$

EXAMPLES

Given to base 10,
$$\log 2 = 0.30103,$$
$$\log 3 = 0.47712,$$
$$\log 5 = 0.69897,$$
$$\log 7 = 0.84509,$$
$$\log 11 = 1.04139.$$

Find the logarithms of the following numbers:

1. 4. 3. 6. 5. 8. 7. 9. 9. 10. 11. 12.

2. 15. 4. 16. 6. $16\frac{2}{3}$. 8. $\frac{1}{128}$. 10. $4\frac{1}{2}$. 12. 75.

13. $\sqrt{1\frac{3}{4}}$. 16. $\sqrt[3]{(24\frac{1}{2})^{-1}}$. 19. .0045. 22. 1.05.

14. $(1.05)^{10}$. 17. 1944. 20. $\sqrt{1.25}$. 23. $\sqrt[3]{.0125}$.

15. $\sqrt[9]{.0045}$. 18. $\sqrt{\frac{4}{343}}$. 21. $\sqrt[3]{15876}$. 24. $\sqrt[4]{.0625}$.

471. The **Characteristic** of a logarithm is the integral part of the logarithm.

ILLUSTRATION. log 487 = 2.68753, in which 2 is the characteristic.

472. The **Mantissa** of a logarithm is the fractional part of the logarithm.

ILLUSTRATION. log 4879 = 3.68833, in which .68833 is the mantissa.

If the characteristic is negative, the sign – is written above it. The sign of the characteristic does not affect the mantissa.

$\bar{2}.98762$ may be written $8-10.98762$, or $8.98762-10$.

473. Prop. 7. The mantissa of the logarithm of a mixed number, or of a decimal fraction, is the same as the mantissa of the number considered as integral.

Dem. The logarithm of 9719 = 3.98762, which means

1. $10^{3.98762} = 9719$. Dividing by 10 successively,
2. $10^{2.98762} = 971.9$ or $\log 971.9 = 2.98762$,
3. $10^{1.98762} = 97.19$ or $\log 97.19 = 1.98762$,
4. $10^{0.98762} = 9.719$ or $\log 9.719 = 0.98762$,
5. $10^{\bar{1}.98762} = 0.9719$ or $\log 0.9719 = \bar{1}.98762$,
6. $10^{\bar{2}.98762} = 0.09719$ or $\log 0.09719 = \bar{2}.98762$,
7. $10^{\bar{3}.98762} = 0.009719$ or $\log 0.009719 = \bar{3}.98762$.

From these cases it is evident that the mantissa remains the same from 9719 to the mixed number 9.719 and to the decimal fraction 0.009719. Hence the truth of the proposition.

The following corollaries also are evident:

474. COR. 1. The characteristic of the logarithm of an integral number, or of a whole number and a decimal fraction, is *one less* than the number of integral places in the number.

ILLUSTRATION. Thus the characteristic of 9719 is 3, or one less than the number of integral places in the number. The characteristic of 9.719 is 0.

475. Cor. 2. The characteristic of a logarithm of a pure decimal fraction is negative and numerically *one greater* than the number of zeros immediately following the decimal point.

Illustrations. The characteristic of 0.9719 is -1, or $\bar{1}$, a negative number numerically one greater than the number of 0's following the decimal point; the characteristic of 0.09719 is $\bar{2}$; of 0.009719 is $\bar{3}$.

EXAMPLES

Given $\log 958.82 = 2.98173$, find:

1. $\log 95.882$.
2. $\log 9.5882$.
3. $\log 9588.2$.

How to Use a Table of Logarithms

476. Given a number, to find its logarithm. Suppose it is required to find log 5576 from specimen pages 389 and 390. 55 is found in the first column on page 390, in the first row. For convenience, the first row of mantissas is placed below.

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474

Model Solution

Find log 5676.

Since $\log 5570 = 3.7459$ and $\log 5580 = 3.7466$,

then $\log 5580 - \log 5570 = 3.7466 - 3.7459 = .0007$.

And $\because$ 5576 is .6 nearer to 5580,

then log 5576 is .6 nearer to log 5580.

$\therefore \log 5576 = \log 5570 + \frac{6}{10}$ of $.0007 = 3.7459 + .00042 = 3.74632$.

In like manner prove that

1. $\log 5528 = 3.74257$; 2. $\log .5504 = \bar{1}.74068$.

477. The process of finding the logarithm of a number having more significant figures than are given in the table is called **Interpolation.**

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9386	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

EXAMPLES

From the table on the two last preceding pages find:

1. $\log 557$, $\log 567$, $\log 573$, $\log 589$.
2. $\log 23$, $\log 457$, $\log 999$, $\log 451$.
3. $\log 765$, $\log (302)^{\frac{1}{3}}$, $\log 400$, $\log (816)^{\frac{1}{2}}$.
4. $\log 5560$, $\log 5678$, $\log 1111$, $\log 6.457$.
5. $\log \sqrt{19.64}$, $\log .006429$, $\log \sqrt{4.96}$, $\log \sqrt[3]{256.8}$.

478. Given the logarithm of a number, to find the number from a table of logarithms. This number is often called **Antilogarithm (antilog)**.

Model Solutions

1. The logarithm of a certain number is 3.7459. Find it.

The mantissa .7459 is found in the row to the right of 55 and under 7. $\therefore$ the number is 557. But the characteristic is 3, which requires 4 integral places. $\therefore$ the number whose logarithm is 3.7459 is 5570.

2. $\log x = 3.7463$. Find the number.

The nearest mantissa lower than .7463 is 7459; the next one above 7463 is 7466.

Since 7463 is 4 above 7459 and 3 below 7466, it is about $\frac{4}{7}$ of the way to 7466.

$\therefore$ antilog 3.7463 is $\frac{4}{7}$ of the way from antilog 3.7459 to antilog 3.7466. Also antilog 3.7463 must be $\frac{4}{7}$ of the way from 557 to 558.

$\therefore$ antilog $3.7463 = 5576$.

EXAMPLES

By the use of the table find the numbers corresponding to the following logarithms:

1. 2.9325.	6. 3.7892.	11. 1.7392.	16. 0.0242.
2. 1.9009.	7. 2.5462.	12. 4.6015.	17. 0.0000.
3. 0.8651.	8. $\bar{3}.2521$.	13. $\bar{3}.3830$.	18. $7.5870 - 10$.
4. $\bar{1}.7774$.	9. 0.4680.	14. 2.6900.	19. $8.6080 - 10$.
5. $\bar{2}.8630$.	10. 0.0350.	15. 0.0050.	20. $9.4261 - 10$.

479. The **Cologarithm (colog)** of a number is the logarithm of its reciprocal.

ILLUSTRATION. $\operatorname{colog} N = \log \frac{1}{N} = \log 1 - \log N = -\log N.$

To make the fractional part of the cologarithm positive, $-\log N$ may be written $10 - \log N - 10$, or $20 - \log N - 20$, or $10k - \log N - 10k$.

MODEL SOLUTIONS

1. $\operatorname{colog} 0.0556 = \log (1 \div 0.0556) = -\log 0.0556$

$$= -\bar{2}.7451 = +2 - .7451 = 1.2549.$$

2. $\operatorname{colog} 543 = (10 - 2.7348) - 10 = 7.2652 - 10 = \bar{3}.2652.$

The cologarithm may be found by subtracting the logarithm from 10 and subtracting 10 from this result.

Instead of subtracting the logarithm of a denominator or a divisor the cologarithm may be added. The object of cologarithms is to make a chain of operations all addition so far as possible.

3. Find the value of $\dfrac{213 \times 57}{7865 \times 0.179}$.

$$\begin{aligned}
\log 213 &= 2.3284 \\
\log 57 &= 1.7559 \\
\operatorname{colog} 7865 &= 6.1043 - 10 \\
\operatorname{colog} 0.179 &= 10.7471 - 10 \\
\hline
&= 20.9357 - 20 \\
&= 0.9357 \\
\operatorname{antilog} 0.9357 &= 8.624.
\end{aligned}$$

$$\therefore \frac{213 \times 57}{7865 \times 0.179} = 8.624.$$

EXAMPLES

1. $\bar{2}.6428 \div 2$.
2. $\bar{4}.5678 \div 5$.
3. $\bar{8}.5643 \div 3$.
4. $\overline{17}.4812 \div 6$.
5. $\sqrt[9]{4782\sqrt[10]{6945}}$.
6. Divide 2764 by 185.
7. Divide 3561 by 7423.
8. Multiply 7462 by 297.
9. Multiply 1257 by 3784.
10. $\sqrt[3]{.56\,\pi}$.
11. $\left(\dfrac{470.6}{9.897}\right)^{-4\frac{1}{2}}$.
12. $\left(\dfrac{\sqrt[3]{47\frac{6}{5}}}{\sqrt[5]{16\frac{1}{7}\,\pi}}\right)^{-\sqrt{\frac{1}{2}}}$.
13. $\sqrt{4.678 \div \pi r^2}$, if $r = \frac{3}{7}$.
14. Divide 15.13 by 1.347.
15. Divide 257.6 by 18.51.
16. $\bar{5}.4735\,(= -6 + 1.4735) \div 3$.

17. $\sqrt{2}$. 18. $\sqrt[3]{3}$. 19. $\sqrt[4]{5}$. 20. $\sqrt[5]{6}$. 21. $\sqrt[12]{1642}$.

22. $\sqrt[3]{.01234}$. 23. $\sqrt{.05432}$. 24. $\sqrt[5]{.0002137}$. 25. $\sqrt[3]{1728}$.

26. $r^{n-1} = 456700$. Find n if $r = 5$.

27. Simplify $\left(\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right)^{-\frac{3}{2}x}$, if $x = .33\frac{1}{3}$.

28. $\dfrac{x}{3.642} = \dfrac{4.642}{.0063}$, to find the value of x.

29. $.123 \div 1.497 \times 2.346 \div 7.423 \times 94.27$.

30. $3333 : 4444 :: 5555 : x$; find x.

31. $66.66 : 7.777 :: .8888 : x$; find x.

32. $\dfrac{20.15}{1.362} = \dfrac{148.6\,x}{24680}$, to find the value of x.

33. In a geometrical ratio, $f = 3$, $l = .0001968$, $r = .3$; find n.

34. In a geometrical ratio, $r = 5$, $l = 1250$, $s = 1562$; find n.

CHAPTER XX

INDETERMINATE EQUATIONS

480. An **Indeterminate Equation** is one which has an unlimited number of solutions. See Art. **332.**

481. An **Indeterminate System** of equations is one which has an unlimited number of solutions.

482. Prop. 1. If the number of independent equations is less than the number of unknown quantities, the solution is in general indeterminate.

Dem. See Art. **334.**

483. Prop. 2. If the number of independent equations is greater than the number of unknown quantities, there is in general no solution, and the system is said to be inconsistent, or impossible.

Dem. See Art. **335.**

484. Prop. 3. Every indeterminate equation of the first degree involving the variables x and y can be reduced to the form $ax + by = c$.

Dem. See Axioms 1 and 2 in Equations. State them.

By restricting the values of the variables to positive integers the number of roots is greatly diminished, even to such an extent that the solution of the equation is possible.

485. An **Impossible Equation** is one that expresses a condition which no solution of the equation will satisfy.

Case 1. If a and b are $+$ and $c -$, the equation $ax+by=c$ is impossible for positive integers.

Dem. Assuming x and y limited to positive values, $ax + by \neq - c$, for the sum of two positive quantities cannot equal a negative quantity.

Case 2. If a, b, c are $+$ and $a + b > c$, the equation $ax + by = c$ is impossible for positive integers.

Dem. Let $x = 1$ and $y = 1$, the smallest positive integers. Then $ax + by = c$ becomes $a + b = c$, which is contrary to the assumption $a + b > c$.

Case 3. If a and b have a common factor not found in c, then $ax + by = c$ is impossible for positive integers.

Dem. Let f be the factor common to a and b.

Then $$\frac{a}{f}x + \frac{b}{f}y = \frac{c}{f}.$$

But $\frac{a}{f}x$ and $\frac{b}{f}y$ are integral and $\frac{c}{f}$ is not.

Since an integer cannot equal a fraction whose numerator is not exactly divisible by the denominator, the equation is impossible.

Model Solutions

1. Solve $3x + 5y = 11$ for positive integral roots.

1. $3x + 5y = 11$, from which

2. $x = \frac{11 - 5y}{3} = 3 - y + \frac{2 - 2y}{3} = 3 - y + \frac{2(1 - y)}{3}.$

3. $5y$ must be < 11, to make x positive.

4. $y < 2\frac{1}{5}$. $\therefore$ y must be less than 3.

Again, to render x integral, $\frac{1-y}{3}$ must be integral, or 0.

Since no value of y less than 3 will render $\frac{1-y}{3}$ integral or 0, except $y = 1$, this is the only value of y which fulfills the conditions.

5. Substituting 1 in place of y in (2), $x = \frac{11-5}{3} = 2$.

6. $\therefore$ $x = 2$ and $y = 1$.

2. Solve $5x + 11y = 254$ for positive integral roots.

1. $5x + 11y = 254$, from which

2. $x = \frac{254 - 11y}{5} = 50 - 2y + \frac{4-y}{5}$.

3. $11y$ must be less than 254.

4. y must be less than $23\frac{1}{11}$.

5. $\therefore$ $y < 24$.

6. $\frac{4-y}{5}$ must be integral.

7. Let $\frac{4-y}{5} = m$.

8. $5m + 4 = y$.

9. If $m = 0$, $y = 4$, and $\therefore$ x will equal 42.

10. If $m = 1$, $y = 9$, and $\therefore$ x will equal 31.

11. If $m = 2$, $y = 14$, and $\therefore$ x will equal 20.

12. If $m = 3$, $y = 19$, and $\therefore$ x will equal 9.

13. If $m = 4$, $y = 24$.

But $y < 24$, so the limit is reached.

$\therefore$ x may equal 42, 31, 20, 9, and y may equal 4, 9, 14, 19.

3. In how many ways can a debt of two dollars be paid with 5-cent and the old-fashioned 3-cent pieces?

Let $x =$ number of 5-cent pieces,

and $y =$ number of 3-cent pieces.

1. Then $5x + 3y = 200$.

2. $x = \frac{200 - 3y}{5} = 40 - \frac{3y}{5}$.

3. $3y < 200$.

4. $y < 67$.

5. $\frac{3y}{5}$ must be integral. $\therefore$ y must be a multiple of 5.

If $y = 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65$,

then $x = 37, 34, 31, 28, 25, 22, 19, 16, 13, 10, 7, 4, 1$.

EXAMPLES

Solve in positive integers:

1. $7x + 9y = 23$.
2. $2x - 4y = 5$.
3. $19x - 117y = 11$.
4. $14x - 17y = -50$.
5. $10x + 12y = -10$.
6. $18x + 17y = 500$.
7. $7x + 10y = 297$.
8. $19x + 5y = 119$.
9. $34x + 89y = 407$.
10. $8y + 5x - 37 = 0$.
11. $58 - 7y = 3x$.
12. $13x - 9 = 17y$.
13. $56x - 75y = 483$.
14. $17x - 49y = -8$.
15. $\begin{cases} x + 3y + 5z = 44, \\ 3x + 5y + 7z = 68. \end{cases}$
16. $\begin{cases} 2x + 11y = 10 + 3z, \\ 3z - 2y + 3x - 30 = 0. \end{cases}$
17. $\begin{cases} 7x - 8y + 9z = 10, \\ 11x + 12y - 13z = 14. \end{cases}$
18. $\begin{cases} 10x + 3y + 2z - 120 = 0, \\ 2x + 3z + 5y = 51. \end{cases}$

CHAPTER XXI

THEORY OF LIMITS

486. An **Independent Variable** is a variable whose value is assumed to be any quantity, however great or small.

487. A **Dependent Variable** is a variable whose value is found from its relation to the independent variable.

ILLUSTRATIONS. Let $y = 3x + 5$.

Assume $x = 2$, then $y = 11$. In this case x is the independent variable and y the dependent variable.

Again, assume $y = 10$, then $x = \frac{5}{3}$. In this case y is the independent variable and x the dependent variable.

488. The **Limit** of a variable is a constant the value of which the variable can never reach, but to which it may approach so nearly by successive steps that the difference shall be less than any assignable quantity.

ILLUSTRATIONS. As .6, .66, .666, $\cdots$ approaches its limit $\frac{2}{3}$, the variable difference between $\frac{2}{3}$ and the successive values of the variable becomes less and less and approaches zero, but never quite reaches zero. The variable difference in this case is $\frac{1}{15}, \frac{1}{150}, \frac{1}{1500}, \cdots$, which becomes very small, but never quite 0.

489. An **Infinitesimal** is a variable whose limit is zero.

490. Notation. The symbol $\doteq$ stands for the expression *approaches as a limit.*

ILLUSTRATION. $x \doteq a$ is read "x approaches a as its limit."

491. Prop. 1. The difference between a variable and its limit approaches zero as the limit.

Dem. If $x \doteq a$, then $x - a \doteq 0$.

For $x - a$ approaches as near to 0 as x does to a, and no nearer.

$$\therefore \text{ if } x \doteq a, \text{ then } x - a \doteq 0.$$

492. Prop. 2. A variable cannot approach two unequal limits at the same time.

Dem. For in approaching as a limit the more remote, the variable would reach a value intermediate between the two, and thereafter, while approaching the one as a limit, would be receding from the other. $\therefore$ the latter cannot be a limit. Hence the truth of the proposition appears.

493. Prop. 3. If one of two variables which are always equal approaches a limit, the other approaches the same limit.

Dem. Let x and y be the two variables which always remain equal, and let $y \doteq a$, then $x \doteq a$.

For if $x = y$, substituting x for y in $y \doteq a$ gives $x \doteq a$.

494. Cor. If two variables are always equal and each approaches a limit, their limits are equal.

Dem. Let $x = y$, $x \doteq a$, $y \doteq b$, then $a = b$.

For $\qquad x - a \doteq 0$, $y - b \doteq 0$, by Prop. 1.

$$\because x = y,\ y - a \doteq 0, \text{ and } y - b \doteq = 0.$$

$$\therefore a = b.$$

495. Prop. 4. The limit of the algebraic sum of a constant and a variable is the algebraic sum of the constant and the limit of the variable.

Dem. Let $x \doteq a$, and let k be any constant.

Then $$k + x \doteq k + a.$$

For if $$x \doteq a, \text{ then } x - a \doteq 0,$$

and $$x - a + k - k \doteq 0.$$

$$\therefore\ x + k \doteq k + a.$$

496. Prop. 5. The limit of the product of a finite constant and a variable equals the product of the finite constant and the limit of the variable.

Dem. Let $x \doteq a$, then $kx \doteq ka$, $k \neq 0$ and being finite.

For if $$x \doteq a, \text{ then } x - a \doteq 0.$$

Since $x - a$ is an infinitesimal,

then $kx - ka$ is an infinitesimal if k is finite.

$$\therefore\ kx - ka \doteq 0, \text{ or}$$

$$kx \doteq ka.$$

497. Prop. 6. If the limit of each of two variables is zero, then the limit of their product is zero.

Dem. Let $x \doteq 0$ and $y \doteq 0$, then $xy \doteq 0$.

For if the differences between x and 0 and y and 0 are both infinitesimal, then the product of x and y must be still nearer to zero.

498. Prop. 7. The limit of the variable sum of two or more variables is equal to the sum of their limits.

Dem. 1. Let $x \doteq a$, $y \doteq b$, $z \doteq c$, etc., to any finite number, as n.

2. Then $x + y + z + \cdots \doteq a + b + c + \cdots$.

3. $x - a \doteq 0$, $y - b \doteq 0$, $z - c \doteq 0, \cdots$.

4. Let $x - a = d$, an infinitesimal.

5. $\quad y - b = d'$, another infinitesimal.

6. $\quad z - c = d''$, another infinitesimal.
etc. etc.

7. Then $(x + y + z + \cdots) - (a + b + c + \cdots)$
$= d + d' + d'' + \cdots$ to n terms.

8. $d + d' + d'' + \cdots =$ an infinitesimal.

9. Let $\quad d + d' + d'' \cdots = D$,

10. and let $\quad d > d' > d'' \cdots$.

11. Then $\quad d < \frac{D}{n}$. Why? Or

12. $\quad nd < D$,

13. $\because$ the variables can be brought so near to their limits as to make $d < \frac{D}{n}$.

14. Then $d + d' + d'' + \cdots < nd$,

15. and $\quad d + d' + d'' + \cdots < D$.

16. $\therefore$ $x + y + z + \cdots$ will approach to $a + b + c + \cdots$ so as to differ from it by less than D; and $\because$ D can be made as small as we please,

17. $\therefore$ $x + y + z + \cdots \doteq a + b + c + \cdots$.

499. Prop. 8. The limit of the variable product of two or more variables is equal to the product of their limits.

Dem. 1. Let $x \doteq a$, $y \doteq b$, $z \doteq c$, $w \doteq e, \cdots$.

2. Then $\text{lt}(xyzw\cdots) = \text{lt}(x) + \text{lt}(y) + \text{lt}(z) + \text{lt}(w) + \cdots$.

3. For $x - a \doteq a$, $y - b \doteq 0$, $z - c \doteq 0, \cdots$.

4. Let $x - a = d$,

5. and $y - b = d'$.

6. Then $\quad x = a + d$,

7. and $\quad y = b + d'$.

8. $\quad xy = ab + ad' + bd + dd'$.

9. $\therefore \text{lt}(xy) = ab + \text{lt}(ad' + bd + dd')$, by Prop. 4.

10. $\quad = ab + a\,\text{lt}(d') + b\,\text{lt}(d) + \text{lt}(dd')$, by Props. 7 and 5.

11. But since $x - a \doteq 0$, and $x - a = d$, then $d \doteq 0$,

12. and since $y - b \doteq 0$, and $y - b = d'$, then $d' \doteq 0$.

13. $\therefore \text{lt}(xy) = ab + a\,\text{lt}(0) + b\,\text{lt}(0) + \text{lt}(0 \cdot 0)$.

14. $\quad = ab$, by Prop. 6.

15. $\quad = \text{lt}(x) + \text{lt}(y)$.

16. $\therefore \text{lt}(xy) = \text{lt}(x) + \text{lt}(y)$.

17. Again $\text{lt}(xyzw \cdots) = \text{lt}(xy) \cdot \text{lt}(zw) \cdots$.

18. $\therefore \text{lt}(xyzw \cdots) = \text{lt}(x) \cdot \text{lt}(y) \cdot \text{lt}(z) \cdot \text{lt}(w) \cdots$.

500. Cor. The limit of the nth power of a variable is equal to the nth power of the limit of the variable.

$\text{Lt}(x)^n = \{\text{lt}(x)\}^n$. Give the demonstration.

501. Prop. 9. The limit of the variable quotient of two variables is equal to the quotient of their limits, provided the limit of the divisor $\neq 0$.

Dem. Let $\quad \dfrac{x}{y} = Q$.

Then $\quad \text{lt}\,Q = \dfrac{\text{lt}(x)}{\text{lt}(y)}$, $\text{lt}(y) \neq 0$.

For $$x = yQ,$$

and $$\text{lt}(x) = \text{lt}(y) \cdot \text{lt}\, Q.$$

$$\therefore\ \text{lt}\, Q \equiv \text{lt}\left(\frac{x}{y}\right) = \frac{\text{lt}(x)}{\text{lt}(y)}.$$

502. Cor. 1. If x be changed to a constant, then

$$\text{lt}\left(\frac{k}{y}\right) = \frac{k}{\text{lt}(y)},\ \text{lt}(y) \neq 0.$$ Why? Prove it.

503. Cor. 2. If $xy = k$, then $\text{lt}(x) \cdot \text{lt}(y) = k$. Prove it.

504. Cor. 3. If $x \div y = k$, then $\text{lt}(x) \div \text{lt}(y) = k$. Prove it if $\text{lt}(y) \neq 0$.

505. Prop. 10. The limit of the sth power of the rth root of a variable is equal to the sth power of the rth root of the limit of the variable; that is

$$\text{lt}\left(x^{\frac{s}{r}}\right) = \{\text{lt}(x)\}^{\frac{s}{r}}.$$

Dem. 1. Let $$x^{\frac{s}{r}} = v,$$

2. then $$v^r = x^s.$$

3. $$\text{lt}(v^r) = \text{lt}(x^s).$$

4. $$\{\text{lt}(v)\}^r = \{\text{lt}(x)\}^s,$$ by Cor., Prop. 8.

5. $$\therefore\ \text{lt}(v) = \{\text{lt}(x)\}^{\frac{s}{r}}.$$

6. But $$\text{lt}(v) = \text{lt}\left(x^{\frac{s}{r}}\right),$$ from (1).

7. $$\therefore\ \text{lt}\left(x^{\frac{s}{r}}\right) = \{\text{lt}(x)\}^{\frac{s}{r}}.$$

EXAMPLES

Find the limit of the following expressions when $x \doteq a$, $y \doteq b$, $z \doteq c$, $v \doteq 3$, $w \doteq 0$:

1. ax.
2. bxy.
3. $ax^{\frac{5}{6}}y^{\frac{2}{3}}z^{\frac{3}{5}}$.
4. $\frac{1}{3}x - \frac{1}{b}y$.
5. $x - y + z$.
6. $x + y + z$.
7. $\frac{a + v + w}{x + y}$.
8. $\frac{x^3}{z^4}$.
9. $\frac{2z^4}{y^7}$.
10. $\frac{vx^2y}{z^3}$.
11. $x^3 + x^2$.
12. $xy^2 - yz^5$.
13. $xz - 5vy$.
14. $ax - by + cz$.
15. $cxy - x^2$.
16. $\frac{\sqrt{xy}}{v^{\frac{2}{3}}} + \frac{3\sqrt[3]{x^2y}}{a\sqrt{yz^3}}$.
17. $\frac{x^3y^{\frac{2}{3}} + ay^{\frac{5}{6}}}{bx^2y^3 - x^3\sqrt{y^3}}$.
18. $\frac{x^2 + ay^3 + pv}{xv^2 + pz^4 + ay^3}$.
19. $x^2y^3 + dx^3y^4 + wv$.
20. $\frac{x + y + z}{x + y} - \frac{x - y}{x(1 - v)}$.
21. $x(x - y) + (x + y)z$.

506. The **Indeterminate Forms** of expressions are as follows:

$$(a)\ \frac{0}{0}, \qquad (b)\ 0 \times \infty, \qquad (c)\ \frac{\infty}{\infty}, \qquad (d)\ \infty - \infty.$$

In a preceding chapter mention was made of the forms $\frac{0}{a}$, $\frac{a}{0}$, and $\frac{0}{0}$. $\frac{0}{a} = ?$ Explain $\frac{a}{0}$. Also $\frac{0}{0}$.

From $\frac{a}{2}$, $\frac{a}{.03}$, $\frac{a}{.0003}$, $\frac{a}{.000003}$, $\cdots$ comes in the limit $\frac{a}{0}$, from which may be derived the following

PRINCIPLE. If the numerator of a fraction is a constant, and the denominator a variable, the value of the fraction increases without limit when the denominator approaches the limit 0. Thus $\left(\frac{a}{x}\right)_{x \doteq 0} = \frac{a}{0} = \infty$, in which ∞ does not represent absolute infinity.

507. The **Fundamental Indeterminate Form** is $\frac{0}{0}$. To this form all others can be reduced.

Dem. 1. $0 \times \infty \equiv 0 \times \frac{a}{0} \equiv \frac{0 \times a}{0} \equiv \frac{0}{0}.$

2. $$\frac{\infty}{\infty} \equiv \frac{\frac{a}{0}}{\frac{b}{0}} \equiv \frac{0}{b} \times \frac{a}{0} \equiv \frac{0}{0}.$$

3. $\infty - \infty \equiv \frac{a}{0} - \frac{b}{0} \equiv \frac{a \cdot 0 - b \cdot 0}{0 \cdot 0} \equiv \frac{0}{0}.$

Model Solutions

1. $$\left\{\frac{x^2 - a^2}{x - a}\right\}_{x \doteq a} = \frac{a^2 - a^2}{a - a} = \frac{0}{0}.$$

But $$\left\{\frac{x^2 - a^2}{x - a}\right\}_{x \doteq a} = \left\{x + a\right\}_{x \doteq a} = a + a = 2a.$$

$$\therefore \frac{x^2 - a^2}{x - a} = 2a \text{ when } x \doteq a.$$

2. $$\left\{\frac{x - a}{x}\right\}_{x \doteq \infty} = \frac{\infty - a}{\infty} = \frac{\infty}{\infty}.$$

But $$\left\{1 - \frac{a}{x}\right\}_{x \doteq \infty} = 1 - \frac{a}{\infty} = 1 - 0 = 1.$$

$$\therefore \left\{\frac{x - a}{x}\right\}_{x \doteq \infty} = 1.$$

3. $$\left\{\frac{5x^3 - 4x^2 - 3}{6x^3 + 7x - 8}\right\}_{x \doteq \infty} = \left\{\frac{5 - \frac{4}{x} - \frac{3}{x^3}}{6 + \frac{7}{x^2} - \frac{8}{x^3}}\right\}_{x \doteq \infty} = \frac{5}{6}.$$

4. $$\left\{\frac{x^2 + 5x - 6}{x^2 - 1}\right\}_{x \doteq 1} = \left\{\frac{(x - 1)(x + 6)}{(x - 1)(x + 1)}\right\}_{x \doteq 1} = \left\{\frac{x + 6}{x + 1}\right\}_{x \doteq 1} = \frac{7}{2}.$$

EXAMPLES

Find the value of:

1. $\left\{\dfrac{x^n-1}{x^n+1}\right\}_{x\doteq 1}$

2. $\left\{\dfrac{ax+b}{bx+a}\right\}_{x\doteq\infty}$

3. $\left\{\dfrac{x^9+a^9}{x^3+a^3}\right\}_{x\doteq -a}$

4. $\left\{\dfrac{x^3+a^3}{x^2-a^2}\right\}_{x\doteq -a}$

5. $\left\{\dfrac{3x^2+4x}{5x^2-6}\right\}_{x\doteq\infty}$

6. $\left\{\dfrac{2x^2-x-21}{x^2-9}\right\}_{x\doteq -3}$

7. $\left\{\dfrac{3x^3+x^4}{2x^2+x^3+3x^4}\right\}_{x\doteq\infty}$

8. $\left\{\dfrac{x^3-8}{x^2-4}\right\}_{x\doteq a}$

9. $\left\{\dfrac{x^5-1}{x-1}\right\}_{x\doteq 1}$

10. $\left\{\dfrac{1-x}{1-bx}\right\}_{x\doteq\infty}$

11. $\left\{\dfrac{2x^2-5x+8}{x-2}\right\}_{x\doteq 3}$

12. $\left\{\dfrac{7x^2-2x-5}{2x^2-7x+3}\right\}_{x\doteq 0}$

13. $\left\{\dfrac{3x^{\frac{1}{4}}+2x^{\frac{2}{3}}+3x^{\frac{1}{2}}}{x^{\frac{1}{4}}+x^{\frac{1}{3}}+x^{\frac{1}{2}}}\right\}_{x\doteq 0}$

14. $\left\{\dfrac{3x^{\frac{1}{4}}+2x^{\frac{2}{3}}+3x^{\frac{1}{2}}}{x^{\frac{1}{4}}+x^{\frac{1}{3}}+x^{\frac{1}{2}}}\right\}_{x\doteq\infty}$

15. $\left\{\dfrac{z^6+z^4+z^2+1}{1-z^4-2z^6-z^2}\right\}_{z\doteq\infty}$

16. $\left\{\dfrac{a^4-2a^2+a+1}{2a^4-3a^3+2a^2-2}\right\}_{a\doteq 0}$

17. $\left\{\dfrac{1+y-3y^3+4y^4}{1+y-y^2-y^3+2y^4}\right\}_{y\doteq\infty}$

18. $\left\{\dfrac{x^4-x^3-9x^2+16x-4}{x^3-2x^2-4x+8}\right\}_{x\doteq 2}$

19. $\left\{\dfrac{x^4+2x^3-3x^2+4x-5}{6x^4-7x^3+8x-9}\right\}_{x\doteq\infty}$

20. $\left\{\dfrac{4x^4+3x^3-2x^2+x-1}{x^4-2x^3+3x^2-4x+5}\right\}_{x\doteq 0}$

CHAPTER XXII

UNDETERMINED COEFFICIENTS

508. Undetermined Coefficients are coefficients assumed in the demonstration of a proposition or the solution of a problem, whose values are to be found by subsequent processes.

509. Prop. If the equation

$$Ax^0 + Bx + Cx^2 + Dx^3 + \cdots = A'x^0 + B'x + C'x^2 + D'x^3 + \cdots$$

is true for all values of the variable x, then the coefficients of the like powers of x in both members are equal.

Dem. 1. Since the equation is true for any value of x, let $x = 0$. Substituting this value of x, the given equation becomes

2. $$A = A'.$$

Subtracting $A (= A')$ from both members of the given equation,

3. $$Bx + Cx^2 + Dx^3 + \cdots = B'x + C'x^2 + Dx^3 + \cdots, \text{ or}$$

4. $$(B - B')x + (C - C')x^2 + (D - D')x^3 + \cdots = 0,$$

5. and $$x[(B - B') + (C - C')x + (D - D')x^2 + \cdots] = 0.$$

6. $$\therefore\ x = 0,$$

7. and $$(B - B') + (C - C')x + (D - D')x^2 + \cdots = 0.$$

8. $$(B - B') = 0, \text{ by letting } x = 0 \text{ in (7)}.$$

9. $$\therefore\ B = B'.$$

By continuing this process it may be proved that

10. $C = C'$,

11. $D = D'$,

etc.

510. Cor. If the equation $Ax^0 + Bx + Cx^2 + Dx^3 + \cdots = 0$ is true for all values of x, then each of the coefficients of $x = 0$.

Dem. Let the student demonstrate this corollary.

Note. The above proposition and corollary apply to equations containing convergent series.

EXPANSION OF FRACTIONS INTO SERIES

Model Solutions

1. Expand $\dfrac{1 + 2x + 0x^2 - 3x^3}{1 - 2x + 3x^2 - 4x^3}$ by undetermined coefficients.

$$\frac{1 + 2x + 0x^2 - 3x^3}{1 - 2x + 3x^2 - 4x^3} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots.$$

Clearing of fractions and collecting terms,

$$1 + 2x - 3x^3 = A + \begin{array}{r|r|r|r|l} B & x + \quad C & x^2 + \quad D & x^3 + \quad E & x^4 + \cdots \\ -2A & -2B & -2C & -2D & \\ & +3A & +3B & +3C & \\ & & -4A & -4B & \end{array}$$

Equating coefficients of like powers of x,

1. $A = 1.$
2. $B - 2A = 2; \quad \therefore B = 4.$
3. $C - 2B + 3A = 0; \quad \therefore C = 5.$
4. $D - 2C + 3B - 4A = -3; \quad \therefore D = -1.$
5. $E - 2D + 3C - 4B = 0; \quad \therefore E = -1.$

$$\therefore \frac{1 + 2x - 3x^3}{1 - 2x + 3x^2 - 4x^3} = 1 + 4x + 5x^2 - x^3 - x^4 + \cdots.$$

2. Expand $\dfrac{3 + 0\,x^2 + x^3}{x^2 + 2\,x^3}$ to 5 terms by undetermined coefficients.

By division it is evident that the first term in the quotient is $\dfrac{3}{x^2}$ or $3\,x^{-2}$.

$$\therefore \; \frac{3 + 0\,x^2 + x^3}{x^2 + 2\,x^3} = Ax^{-2} + Bx^{-1} + Cx^0 + Dx + Ex^2 + \cdots.$$

Clearing of fractions and collecting terms,

$$3 + 0\,x^2 + x^3 = Ax^0 + \left.\begin{array}{r} B \\ +\,2\,A \end{array}\right| x + \left.\begin{array}{r} C \\ +\,2\,B \end{array}\right| x^2 + \left.\begin{array}{r} D \\ +\,2\,C \end{array}\right| x^3 + \left.\begin{array}{r} E \\ +\,2\,D \end{array}\right| x^4 + \cdots$$

Equating the coefficients of like powers of x,

1. $A = 3.$
2. $B + 2\,A = 0; \; \therefore \; B = -\,6.$
3. $C + 2\,B = 0; \; \therefore \; C = +\,12.$
4. $D + 2\,C = 1; \; \therefore \; D = -\,23.$
5. $E + 2\,D = 0; \; \therefore \; E = +\,46.$

$$\therefore \; \frac{3 + x^3}{x^2 + 2\,x^3} = 3\,x^{-2} - 6\,x^{-1} + 12 - 23\,x + 46\,x^2 - \cdots.$$

Prove this answer by long division.

EXAMPLES

Expand to 5 terms and prove the result by division:

1. $\dfrac{1 + 12\,x}{1 - 3\,x^2}.$

2. $\dfrac{1 + 2\,x^2 + 3\,x^4}{1 - 2\,x^2 - x^4}.$

3. $\dfrac{x^4 - 3\,x^2 + 2}{x^2 - 5\,x + 6}.$

4. $\dfrac{3 + x}{2 - 7\,x - x^2}.$

5. $\dfrac{1 + 3\,x}{2 + x + x^2}.$

6. $\dfrac{1 - 2\,x + 3\,x^2}{1 + 2\,x + 3\,x^2}.$

7. $\dfrac{1 + x + 2\,x^2}{x^2 + 2\,x^3 + 2\,x^4}.$

8. $\dfrac{1}{x^2 - 2\,x^3 + x^4}.$

9. $\dfrac{x - 2\,x' + 3\,x^4}{x^3 + x^4 - 3\,x^5}.$

10. $\dfrac{2\,x - 3\,x^2 + 4\,x^3}{1 + 3\,x - 4\,x^2}.$

11. $\dfrac{1 - y + 2\,y^2 + 3\,y^3}{2\,y - 3\,y^2 + 4\,y^3}.$

12. $\dfrac{z - 2\,z^2 + 5\,z^3}{1 + 3\,z - 5\,z^2}.$

EXPANSION OF SURDS INTO SERIES

Model Solutions

1. Expand $\sqrt{1+2x}$ to 5 terms.

1. $\sqrt{1+2x} = (1+2x)^{\frac{1}{2}} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots.$

2. $\therefore\ 1 + 2x = A^2 + B^2x^2 + C^2x^4 + D^2x^6 + E^2x^8 + 2ABx$
$\quad + 2ACx^2 + 2ADx^3 + 2AEx^4 + 2BCx^3 + 2BDx^4$
$\quad + 2BEx^5 + 2CDx^5 + 2CEx^6 + 2DEx^7 + \cdots.$

3. $= A^2 + 2ABx + (B^2 + 2AC)x^2 + (2AD + 2BC)x^3$
$\quad + (C^2 + 2AE + 2BD)x^4 + (2BE + 2CD + 2AF)x^5 +$
etc. etc.

4. $\therefore \qquad A^2 = 1, \text{ or } A = 1.$

5. $\qquad 2AB = 2, \text{ or } B = 1.$

6. $\qquad B^2 + 2AC = 0, \text{ or } C = -\frac{1}{2}.$

7. $\qquad 2AD + 2BC = 0, \text{ or } D = \frac{1}{2}.$

8. $C^2 + 2AE + 2BD = 0, \text{ or } E = -\frac{5}{8}.$
etc. etc.

9. $\therefore\ \sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 + \cdots.$

2. Expand $(a^2+x^2)^{\frac{3}{2}}$ to 7 terms.

Let $(a^2+x^2)^{\frac{3}{2}} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \cdots.$

Then

$$
\begin{array}{r|r|r|r|r|r|l}
a^6+3a^4x^2+3a^2x^4+x^6 = A^2+AB & x+AC & x^2+AD & x^3+AE & x^4+AF & x^5+AG & x^6+\cdots \\
+AB & +B^2 & +BC & +BD & +BE & +BF & \\
 & +AC & +BC & +C^2 & +CD & +CE & \\
 & & +AD & +BD & +CD & +D^2 & \\
 & & & +AE & +BE & +CE & \\
 & & & & +AF & +BF & \\
 & & & & & +AG &
\end{array}
$$

1. $\therefore A^2 = a^6, \quad \text{or} \quad A = a^3.$

2. $2AB = 0, \quad \text{or} \quad B = 0.$

3. $2AC + B^2 = 3a^4, \quad \text{or} \quad C = \frac{3}{2}a.$

4. $2AD + 2BC = 0, \quad \text{or} \quad D = 0.$

5. $2AE + 2BD + C^2 = 3a^2, \quad \text{or} \quad E = \frac{3}{8a}.$

6. $2AF + 2BE + 2CD = 0, \quad \text{or} \quad F = 0.$

7. $2AG + 2BF + 2CE + D^2 = 1, \quad \text{or} \quad G = -\frac{1}{16a^3}.$

8. $$\therefore (a^2 + x^2)^{\frac{3}{2}} = a^3 + 0x + \frac{3}{2}ax^2 + 0x^3 + \frac{3}{8a}x^4 + 0x^5 - \frac{1}{16a^3}x^6 + \cdots.$$

EXAMPLES

Expand to five terms:

1. $\sqrt{1 - x^2}$.
2. $\sqrt{2 + 3x}$.
3. $\sqrt[3]{(1 + x)^2}$.
4. $\sqrt[3]{1 - x}$.
5. $(a - y)^{\frac{3}{2}}$.
6. $\sqrt{(a^2 - x^2)^3}$.
7. $\sqrt{1 - 2x - 3x^2}$.
8. $5(2 - 3x + 4x^2)^{\frac{1}{2}}$.
9. $\frac{2}{3}\sqrt{\frac{1}{2} - \frac{2}{3}x + \frac{3}{4}x^2}$.

DECOMPOSITION OF FRACTIONS

511. In the following cases **the degree of the denominator** $>$ the degree of the numerator:

Case 1. The denominator is composed of real and *unequal* factors of the *first* degree.

Model Solution

Resolve $\dfrac{2x + 1}{6x^2 - 5x - 6}$ into partial fractions.

1. Let $\dfrac{2x + 1}{6x^2 - 5x - 6} = \dfrac{A}{2x - 3} + \dfrac{B}{3x + 2}.$

2. Then $2x+1=3Ax+2A+2Bx-3B.$

3. $1+2x=2A-3B+(3A+2B)x.$

4. $\therefore 2A-3B=1$

5. and $3A+2B=2$

6. $4A-6B=2$

7. $9A+6B=6$

$13A=8$

8. $A=\frac{8}{13}.$

9. $B=\frac{1}{13}.$

10. $\therefore \dfrac{2x+1}{6x^2-5x-6}=\dfrac{\frac{8}{13}}{2x-3}+\dfrac{\frac{1}{13}}{3x+2}=\dfrac{8}{26x-39}+\dfrac{1}{39x+26}.$

EXAMPLES

1. $\dfrac{x}{x^2-5x+6}.$

2. $\dfrac{1-x}{3x^2-2x-1}.$

3. $\dfrac{x-4}{6x^2-x-1}.$

4. $\dfrac{2a+3}{a^2+11a+30}.$

5. $\dfrac{3a-5}{a^2-a-6}.$

6. $\dfrac{2y+7}{y^2-5y-6}.$

7. $\dfrac{2-x}{x(x+1)(x-3)}.$

8. $\dfrac{x^2+2x-1}{(x-1)(x-2)(x-3)}.$

9. $\dfrac{3x^2+4x-5}{x^3+6x^2+3x-10}.$

10. $\dfrac{6x^2-7x+8}{x^3-11x^2+36x-36}.$

Case 2. The denominator is composed of real and *equal* factors of the *first* degree.

Model Solution

Resolve $\dfrac{a^2-2ax+3x^2}{x^3-3x^2a+3xa^2-a^3}$ into partial fractions.

1. Let $\dfrac{a^2-2ax+3x^2}{x^3-3x^2a+3xa^2-a^3}=\dfrac{A}{(x-a)^3}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-a)}.$

2. Then $a^2-2ax+3x^2=A+Bx-Ba+Cx^2-2Cxa+Ca^2.$

3. $A - Ba + Ca^2 = a^2,$

4. $B - 2\,Ca = -2\,a,$

5. and $C = 3.$

6. $B = 4\,a.$

7. $A = 2\,a^2.$

8. $\therefore\ \dfrac{a^2 - 2\,ax + 3\,x^2}{x^3 - 3\,x^2a + 3\,xa^2 - a^3} = \dfrac{2\,a^2}{(x-a)^3} + \dfrac{4\,a}{(x-a)^2} + \dfrac{3}{x-a}.$

EXAMPLES

1. $\dfrac{3\,x - 2}{(x-3)^2}.$

2. $\dfrac{x}{x^2 + 2\,x + 1}.$

3. $\dfrac{2\,x^2 - 3\,x + 4}{(x-1)^3}.$

4. $\dfrac{x+3}{x^2 - 10\,x + 25}.$

5. $\dfrac{a-2}{a^2 + 16\,a + 64}.$

6. $\dfrac{2\,x - 5}{4\,x^2 - 12\,x + 9}.$

7. $\dfrac{x^2 - 2\,x + 3}{x^3}.$

8. $\dfrac{x^2 + 7\,x + 6}{(x+1)^3}.$

9. $\dfrac{1-x}{x^2 - 18\,x + 81}.$

10. $\dfrac{7-x}{9\,x^2 - 12\,x + 4}.$

11. $\dfrac{2\,x + 3\,a}{16\,a^2 + 24\,ax + 9\,x^2}.$

12. $\dfrac{a^2 + 2\,ax - x^2}{x^3 + 3\,x^2a + 3\,xa^2 + a^3}.$

Case 3. The denominator is composed of real and *equal* factors of the *second* degree.

Model Solution

Resolve $\dfrac{3 - 2\,x + x^2}{x^4 + 2\,x^2 + 1}$ into partial fractions.

1. Let $\dfrac{3 - 2\,x + x^2}{(x^2+1)^2} = \dfrac{Ax + B}{(x^2+1)^2} + \dfrac{Cx + D}{x^2 + 1}.$

2. $\therefore\ 3 - 2\,x + x^2 = Ax + B + Cx^3 + Dx^2 + Cx + D,$ or

3. $3 - 2\,x + x^2 = B + D + (A + C)x + Dx^2 + Cx^3.$

4. $\therefore\ B + D = 3,$

5. $A + C = -2,$

6. $D = 1,$

7. $C = 0,$

8. $A = -2,\ B = 2,\ C = 0,\ D = 1.$

9. $$\therefore\ \frac{3 - 2x + x^2}{x^4 + 2x^2 + 1} = \frac{1}{x^2 + 1} - \frac{2(x-1)}{(x^2+1)^2}.$$

EXAMPLES

1. $\dfrac{2x - 3}{x^4 - 2x^2 + 1}.$

2. $\dfrac{5x^3 + 2x - 3}{x^4 - 4x^2 + 4}.$

3. $\dfrac{6x^2 - 7x + 8}{x^4 + 6x^2 + 9}.$

4. $\dfrac{1 + 2x - 3x^2 + 4x^3}{(x^2 + x + 1)^2}.$

5. $\dfrac{2 + 3x - 4x^2}{(x^2 - x + 2)^2}.$

6. $\dfrac{5x^3 - 4x^2 + 3x - 2}{(x^2 - x + 1)^2}.$

Case 4. The denominator is composed of factors of the *first* degree, *some* of which are *equal.*

Model Solution

Resolve $\dfrac{5 + 6x - 7x^2}{x^2(1-x)(2-x)^2}$ into partial fractions.

Let $$\frac{5 + 6x - 7x^2}{x^2(1-x)(2-x)^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1-x} + \frac{D}{(2-x)^2} + \frac{E}{(2-x)}.$$

$$5 + 6x - 7x^2 = \begin{array}{lr|r|r|r|l} 4A & -8A & x + 5A & x^2 - A & x^3 - B & x^4 \\ & +4B & -8B & +5B & +C & \\ & & +4C & -4C & +E & \\ & & +D & -D & & \\ & & +2E & -3E & & \end{array}$$

1. $4A = 5.$

2. $-8A + 4B = 6.$

3. $5A - 8B + 4C + D + 2E = -7.$

4. $-A + 5B - 4C - D - 3E = 0.$

5. $-B + C + E = 0.$

6. $A=\frac{5}{4}$, from (1).

7. $B=4$, from (2) and (6).

8. $E=0$, from (3) + (4) and (6), (7).

9. $C=4$, from (5).

10. $D=\frac{11}{4}$, from (4).

11. $$\therefore \frac{5+6x-7x^2}{x^2(1-x)(2-x)^2}=\frac{5}{4x^2}+\frac{4}{x}+\frac{4}{1-x}+\frac{11}{4(2-x)^2}.$$

EXAMPLES

1. $\dfrac{4+x}{(x+2)(x-1)^2}$.

2. $\dfrac{5x-7x^3}{(x+1)^2(x-2)^2}$.

3. $\dfrac{2x^3-4x^2-7x+2}{x(x+1)(x-2)^2}$.

4. $\dfrac{2-7x+12x^2}{(x-3)(x-3)(x-7)x}$.

5. $\dfrac{3x^2-2x^3+3}{x^2(x+3)(x-5)(x-5)}$.

6. $\dfrac{x^2}{(x-1)^3(x+1)}$.

7. $\dfrac{7x-5}{x(2-x)^2(x-2)}$.

8. $\dfrac{2x-1}{(x^2-8x+16)(2-x)}$.

9. $\dfrac{11x-x^2}{x^3(x^2+4x+4)(1-x)}$.

10. $\dfrac{2x^4-3x^3+4x^2-5x+6}{x^2(x-1)^2(1-x)^2}$.

Case 5. The denominator is composed of *equal* or *unequal* factors, *some* of which are of the *second* degree.

Model Solution

Resolve $\dfrac{3-2x+x^3}{(x-1)^2(x^2+1)}$ into partial fractions.

Let $$\frac{3-2x+x^3}{(x-1)^2(x^2+1)}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}.$$

$$3-2x+x^3 = A-B+D \begin{array}{r|r|r|l} +B & x+A & x^2+B & x^3 \\ +C & -B & +C & \\ -2D & -2C & & \\ & +D & & \end{array}$$

$$
\begin{array}{lr}
1. & A - B + D = 3. \\
2. & B + C - 2D = -2. \\
3. & A - B - 2C + D = 0. \\
4. & B + C = 1. \\
\hline
5. & D = \frac{3}{2}, \\
 & C = \frac{3}{2}, \\
 & B = -\frac{1}{2}, \\
 & A = 1.
\end{array}
$$

$$\therefore \frac{3 - 2x + x^3}{(x-1)^2(x^2+1)} = \frac{1}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{3x+3}{2(x^2+1)}.$$

EXAMPLES

1. $\dfrac{x}{(x^2+1)(x+2)}$.

2. $\dfrac{3x-4}{(x-1)(x^2+2)}$.

3. $\dfrac{ax + a^2x^2 + a^3x^3}{a^4 + 4x^4}$.

4. $\dfrac{x^2+x+1}{(x-7)(x^2-x+1)}$.

5. $\dfrac{1 - 2x + 3x^2 + 4x^3}{(x^2+3)(x^2-3)}$.

6. $\dfrac{3x - 4x^3}{(x^2+x+1)(x^2-x+1)}$.

7. $\dfrac{2x-17}{(x-5)(x^2+1)^2}$.

8. $\dfrac{ax + a^2x^2 + a^3x^3}{x^4 + x^2a^2 + a^4}$.

9. $\dfrac{(1-x)(1+x-x^2)}{4x^4+1}$.

10. $\dfrac{2x^5 + 3x^3 - 4x - 3}{x^6 - 1}$.

11. $\dfrac{(x-1)(x+1)(x-7)}{x^6 - 64}$.

12. $\dfrac{x^3 - 3x^2 + 5}{x(x+a)^3(x-b)(x^2+a^2)}$.

512. In the following case **the degree of the denominator** $\leqq$ the degree of the numerator:

This case can be brought under the preceding cases by reducing the improper fraction to a mixed form and then decomposing the fractional part in the usual way.

MODEL SOLUTION

Resolve $\dfrac{2x^2 - 3x + 7}{3x^2 + 5x + 2}$ into partial fractions.

$$\frac{2x^2 - 3x + 7}{3x^2 + 5x + 2} = \frac{2}{3} - \frac{\frac{19}{3}x - \frac{17}{3}}{3x^2 + 5x + 2}.$$

1. Let $\dfrac{\frac{19x}{3} - \frac{17}{3}}{3x^2 + 5x + 2} = \dfrac{A}{x+1} + \dfrac{B}{3x+2}.$

2. $\frac{19}{3}x - \frac{17}{3} = 2A + B + (3A + B)x.$

3. $2A + B = -\frac{17}{3}.$

4. $3A + B = \frac{19}{3}.$

5. $A = 12.$

6. $B = -\frac{89}{3}.$

$$\therefore \frac{2x^2 - 3x + 7}{3x^2 + 5x + 2} = \frac{2}{3} - \frac{12}{x+1} - \frac{-\frac{89}{3}}{3x+2} = \frac{2}{3} - \frac{12}{x+1} + \frac{89}{9x+6}.$$

EXAMPLES

1. $\dfrac{x^7}{x^6 - 1}.$

2. $\dfrac{2x^4 - 15}{1 + 4x^4}.$

3. $\dfrac{4x^2 - 3x + 7}{x^2 - 3x + 2}.$

4. $\dfrac{x^5 + x^4 + 5x + 7}{x^4 + x^2 + 1}.$

5. $\dfrac{3x^3 - 4x^2 - 7x + 3}{x^2 - 4x + 4}.$

6. $\dfrac{8 - 7x + 6x^2 - 5x^3}{6x^2 + 5x - 6}.$

7. $\dfrac{x^8}{1 - 4x^4}.$

8. $\dfrac{8x^4 - 4x^2 + 5}{4x^3 - 4x^2}.$

9. $\dfrac{15x^3 - 4x^2 + 7}{(x^2 + 1)(x + 3)}.$

10. $\dfrac{4x^3 + 5x^2 + 7x + 9}{(3x^2 - 5x + 6)(x - 1)}.$

11. $\dfrac{x^4 - 7x^3 + 4x - 3x^2 + 4}{(x + 1)^2(x - 3)}.$

12. $\dfrac{7 + 6x - 5x^2 + 4x^3 - 3x^4}{(1 - x)^2(1 + x)}.$

CHAPTER XXIII

PERMUTATIONS AND COMBINATIONS

513. Permute signifies to rearrange. **Permutation** signifies the arrangement of a definite number of things in all possible ways, one after another. **Combination** signifies any one of such possible arrangements.

514. Permutations are the different orders in which a number of things can be arranged.

ILLUSTRATIONS. 1. The permutations of the letters a and b are

$$ab, \; ba.$$

2. The permutations of the three letters abc taken two at a time are

$$ab, \; ba; \; ac, \; ca; \; bc, \; cb.$$

3. The permutations of the letters abc taken three at a time are

$$abc, \; acb; \; bac, \; bca; \; cab, \; cba.$$

515. The **permutations of n things taken r at a time** are the different groups of r things taken from n different things and arranged in every possible order.

In Illustration 2 above, n things are represented by the three letters abc, and r at a time by two at a time. The different groups of the three letters taken two at a time are

$$ab, \; ba, \; ac, \; ca, \; bc, \; cb,$$

the letters being arranged in every possible order.

516. Notation ^nP_r. The number of permutations of n different things taken r at a time is represented by the symbol ^nP_r. Read $^5P_3$; $^{10}P_7$; ^xP_y.

MODEL SOLUTION

Show by the use of the letters a, b, c, d, that $^4P_2 = 12$.

ab, *ba*; *ac*, *ca*; *ad*, *da*; *bc*, *cb*; *bd*, *db*; *cd*, *dc*, or 12 in all.

EXAMPLES

1. (*a*) Show that $^3P_2=6$; (*b*) $^4P_4=?$; (*c*) $^6P_1=?$; (*d*) $^4P_3=?$
2. (*a*) Show that $^nP_1=n$; (*b*) $^3P_1=?$; (*c*) $^4P_2=?$; (*d*) $^3P_3=?$
3. (*a*) Write out the permutations of the letters of the name *John* taken two at a time; (*b*) of the name *Ada* taken two at a time, the first A remaining capitalized.

517. Prop. 1. If one thing can be done in m ways, and after it has been done in any one of these ways, a second thing can be done in n ways, the two things can be done in mn ways.

ILLUSTRATIONS. 1. In how many ways can scholarships be awarded to seven candidates for a mathematical, and four for a classical, scholarship?

a. A scholarship can be given to the first mathematical candidate and to any one of the four classical candidates, making four ways.

b. A scholarship can be given to the second mathematical candidate and to any one of the four classical candidates, making four ways.

c. Since there are seven mathematical candidates it is evident that there will be seven of these four ways, or twenty-eight ways in all.

2. Five ferryboats ply between two cities, A and B. In how many ways can a man go from A to B and return by a different boat?

He can make the first trip in five ways. With each of the five ways he has the choice of four ways in which to return, or $4 \times 5 = 20$ ways in all.

3. In how many ways can I give three golf balls to seven caddies, without giving more than one to the same boy?

The first golf ball can be given in seven ways. With each of these seven ways the second golf ball can be given in six ways to the remaining six boys who have no golf balls, or $7 \times 6 = 42$ ways. With each of these forty-two ways, the third golf ball can be given in five ways, or $42 \times 5 = 210$ in all.

Dem. After the first thing has been done in any one of m ways, the second thing can be done in n different ways and associated with the one way the first was done. Hence there are n ways of doing the two operations by considering *one* way of doing the first. By considering m ways of doing the first associated with the n ways of doing the second, there will be $n \times m$ ways, or mn ways of doing the two things.

518. Prop. 2. $^np_r = n(n-1)(n-2)(n-3) \cdots (n-r+1)$.

Dem.

1. $^np_2 = n(n-1)$.

For, if there are r things to be taken there must be r places to fill.

The first place can be filled in any one of the n ways, and after this has been filled, the second place can be filled in $(n-1)$ ways. But by Prop. 1 if the first place can be filled

in n ways, and the second in $(n-1)$ ways, the two places can be filled in $(n-1)$ times n ways, or $n(n-1)$ ways.

2. ${}^np_3 = n(n-1)(n-2)$,

Because for every way of filling the first two places, there are $n-2$ ways of filling the first, second, and third.

3. ${}^np_4 = n(n-1)(n-2)(n-3)$.

It is readily seen that the last binomial in each case is n minus a number one less than the value of r in ${}^np_r$. $\therefore$ the general expression for the last factor is $[n-(r-1)]$, or $(n-r+1)$.

4. $\therefore\ {}^np_r = n(n-1)(n-2)(n-3)\cdots(n-r+1)$.

519. Cor. ${}^np_n = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1 = n!$ or $\underline{|n}$ (read "factorial n").

Dem. For (4) above becomes ${}^np_n = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$, when $r=n$. $\therefore\ {}^np_n = \underline{|n}$.

If the second member of ${}^np_r = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ be both multiplied and divided by $\underline{|n-r}$,

$$ {}^np_r = \frac{\{n(n-1)(n-2)\cdots(n-r+1)\}\{(n-r)(n-r-1)\cdots 2\cdot 1\}}{(n-r)(n-r-1)\cdots 2\cdot 1} $$

$$ = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)(n-r-2)\cdots 3\cdot 2\cdot 1} $$

$$ = \frac{\underline{|n}}{\underline{|n-r}}. $$

Model Solutions

1. ${}^8p_7 = ?$

$$ {}^np_r = {}^8p_7 = 8(8-1)(8-2)\cdots(8-7+1) $$

$$ = 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2 = 40320. $$

2. How many different arrangements can be made by taking 5 of the letters of the word *equation?*

$$^{n}p_r = {}^{8}p_5 = \frac{\lfloor n}{\lfloor n-r} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1} = 8\cdot7\cdot6\cdot5\cdot4 = 6720.$$

3. In how many ways can 5 boys sit in the front row of seats of a schoolroom?

$$^{n}p_n = {}^{5}p_5 = \lfloor 5 = 5\cdot4\cdot3\cdot2\cdot1 = 120.$$

EXAMPLES

1. Find the value of $^{10}p_3$.

2. Find the value of $^{19}p_5$.

3. Find the value of $^{7}p_7$.

4. Find the value of $^{11}p_8$.

5. How many arrangements can be made of the letters of the word *word?* *Pencil?* *Slate?* *Logarithms?*

6. How many numbers can be represented by the 9 digits 1, 2, 3 ⋯ 9, taken 2 at a time? 3 at a time? 4 at a time?

7. In how many ways can 10 gentlemen and 10 ladies arrange themselves into couples?

8. How many different signals can be made with 12 different colored flags by displaying them (*a*) 2 at a time; (*b*) 3 at a time; (*c*) 4 at a time; (*d*) all at a time?

9. How many changes can be rung with a peal of 5 bells?

10. There are 10 candidates for a classical, 8 for a mathematical, and 6 for a scientific scholarship. In how many ways can the scholarships be awarded?

520. Prop. 3. The number of permutations of n things taken all at a time, of which p things are alike, is

$$\frac{\lfloor n}{\lfloor p}.$$

Dem. If the letters a, c, f of *abcdef* are permuted while b, d, e remain fixed in position, the number of permutations will be ${}^3p_3 = \underline{|3}$. If a, c, f become alike, say a, a, a, there will be but *one* permutation. $\therefore$ there are $\underline{|3}$ times as many permutations when no letters are alike as there are when 3 letters are alike. Hence the number of permutations when 3 are alike is as many as $\underline{|3}$ is contained times in $\underline{|n}$, or

$$\frac{\underline{|n}}{\underline{|3}};$$

when 4 are alike, $$\frac{\underline{|n}}{\underline{|4}};$$

and when p are alike, $$\frac{\underline{|n}}{\underline{|p}}.$$

521. Cor. 1. The number of permutations of n things taken all at a time, of which p things are alike, q others alike, and r others alike, is

$$\frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}.$$

Dem. The number of permutations when q are alike is

$$\frac{\underline{|n}}{\underline{|q}}, \text{ by Prop. 3;}$$

when r are alike, $$\frac{\underline{|n}}{\underline{|r}}, \text{ by Prop. 3;}$$

$\therefore$ when p, q, r things are alike the number is

$$\frac{\underline{|n}}{\underline{|p} \cdot \underline{|q} \cdot \underline{|r}}.$$

Model Solutions

Find the number of permutations of the letters in (*a*) Mississippi; (*b*) Algebra; (*c*) Institutions.

$$(a)\quad \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r} = \frac{\lfloor 11}{\lfloor 4 \lfloor 4 \lfloor 2} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1\cdot 4\cdot 3\cdot 2\cdot 1\cdot 2\cdot 1} = 34650,$$

in which $n = 11$, $p = s\cdot s\cdot s\cdot s = 4$, $q = i\cdot i\cdot i\cdot i = 4$, $r = p\cdot p = 2$.

$$(b)\quad \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r} = \frac{\lfloor 7}{\lfloor 2} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1} = 2520,$$

in which $n = 7$, $p = a\cdot a = 2$.

$$(c)\quad \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r} = \frac{\lfloor 12}{\lfloor 3 \lfloor 2 \lfloor 2 \lfloor 3} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 3\cdot 2\cdot 1} = 3326400.$$

EXAMPLES

Find the number of permutations that can be made out of the letters of the words:

1. (*a*) Permutations; (*b*) Ecclesiastical; (*c*) Possessions.

2. (*a*) United States; (*b*) Ann Arbor; (*c*) Independence.

3. With 3 consonants and 3 vowels, how many words of 6 letters can be formed, each word beginning with a consonant and ending with a vowel?

4. In how many ways can 2 sixes, 3 fives, and 5 twos be thrown with 10 dice?

522. Prop. 4. ${}^nP_r = n^r$ when each of the n things may be repeated.

Dem. ${}^nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+1)$.

Since repetitions are allowed, the second can be filled in n ways, the third in n ways, etc.

Therefore ${}^nP_r = n\cdot n\cdot n\cdot n\cdots$, or r factors of n.

Therefore ${}^nP_r = n^r$.

Model Solution

Find ${}^5P_3$ when repetitions are allowed.

$${}^nP_r = {}^5P_3 = 5^3 = 125.$$

EXAMPLES

1. Find ${}^6P_3$ when repetitions are allowed.

2. Find ${}^8P_4$ when repetitions are allowed.

3. Find ${}^7P_5$ when repetitions are allowed.

4. Find ${}^{10}P_3$ when repetitions are allowed.

5. How many numbers can be formed from the digits 5, 6, 7, 8, repetitions being allowed?

6. What is the chance of guessing correctly the three numbers on which a combination lock of 50 numbers is set? A lock of 10 numbers? Of 100 numbers? Of 27 numbers?

523. Combinations are the different groups which can be formed from a given number of things when the order of arrangement is not taken into account.

Illustration. The combinations of a, b, c, d, taken 2 at a time, are

$$ab,\ ac,\ ad,\ bc,\ bd,\ cd,$$

each of which presents a different group of two letters.

From this it is seen that the *number* of things in each group is taken into account and not the order of their arrangement.

524. The **combinations of n things taken r at a time** are the different groups of r things taken from n different things when the order of arrangement is not taken into account.

525. Notation ^nC_r. The number of combinations of n different things taken r at a time is represented by the symbol ^nC_r. Read $^8C_2$; $^{10}C_4$; ^xC_y.

EXAMPLES

1. Show that $^4C_2 = 6$.
2. Show that $^3C_3 = 1$.
3. Show that $^nC_n = 1$.
4. Show that $^4C_3 = 4$.
5. Find the value of $^3C_2$; $^5C_2$; $^5C_3$.

526. Prop. 5. $$^nC_r = \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{\underline{|r}}.$$

Dem. — 1. $^nP_r = {}^nC_r \times \underline{|r}$, because each of the combinations in ^nC_r contains a group of r different things which can be arranged among themselves in $\underline{|r}$ ways.

2. $\because\ {}^nC_r \times \underline{|r} = {}^nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+1)$,

3. $$\therefore\ {}^nC_r = \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{\underline{|r}}.$$

527. Cor. 1. $$^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

Dem. — 1. $$^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)\underline{|n-r}}{\underline{|r}\,\underline{|n-r}}.$$

2. $$^nC_r = \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{\underline{|r}\,\underline{|n-r}}.$$

3. $$^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

528. Cor. 2. $$^nC_r = \frac{^nP_r}{\underline{|r}}$$

529. COR. 3. $\quad {}^nC_r \equiv {}^nC_{n-r}.$

Dem. $\quad$ 1. $\quad {}^nC_r = \dfrac{\lfloor n}{\lfloor r \lfloor n-r}.$

Substitute $n-r$ for r in (1), and

2. $\quad {}^nC_{n-r} = \dfrac{\lfloor n}{\lfloor n-r \lfloor r}.$

3. $\quad \therefore {}^nC_r \equiv {}^nC_{n-r}.$

ILLUSTRATION. $\quad {}^{25}C_{23} = {}^{25}C_2 = \dfrac{25 \times 24}{2 \times 1} = 300.$

EXAMPLES

1. Find the value of ${}^{15}C_{14}$; ${}^{12}P_{10}$; ${}^5P_5$; ${}^6C_6$; ${}^{47}C_{45}$.

2. Find the value of ${}^{20}C_{15}$; ${}^7C_3$; ${}^{10}C_3$; ${}^nC_n$.

3. How many different permutations can be made out of the letters of the word *assassination*, taken all at a time?

4. If ${}^nC_2 = 21$, find n.

5. If ${}^{10}C_r = 45$, find r.

6. How many parties of 10 men can be formed from a company of 20 men?

7. How many different whist parties of 4 each can be made out of 6 players?

8. If the number of permutations of 12 things taken r at a time is 42 times the number taken $r-2$ at a time, find r.

9. Out of 8 letters, x of which are alike, 336 words can be formed. Find the value of x.

10. How many different permutations can be formed from the letters of the word *Cincinnati? Ethel? Involution? Constitution? Shakespeare? Louise? Elizabeth? Nancy?*

11. How many committees, each containing 3 Republicans and 4 Democrats, can be made up from 12 Republicans and 16 Democrats?

12. Find the number of permutations of 10 letters taken all at a time, when there are 2 a's, 3 b's, and 5 c's.

CHAPTER XXIV

DETERMINANTS

SECOND ORDER

530. By the Rule of Cross-Multiplication under Problem 5 of Chapter X, the following system of equations may be solved in the manner shown below:

Let $\left\{\begin{aligned} ax + by + c &= 0, \\ a'x + b'y + c' &= 0 \end{aligned}\right\}$ be two simultaneous linear equations.

$$\begin{matrix} b & & c & & a & & b \\ & \times & & \times & & \times & \\ b' & & c' & & a' & & b' \end{matrix}$$

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{1}{ab' - a'b},$$

$$\therefore\ x = \frac{bc' - b'c}{ab' - a'b},$$

and

$$y = \frac{ca' - c'a}{ab' - a'b}.$$

These results can be obtained directly from the coefficients when written in the regular order as follows:

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \end{vmatrix}.$$

The difference between the first diagonal products, $ab' - a'b$, is the denominator of each fractional root; the difference between the second diagonal products, $bc' - b'c$, is the numerator of the fraction representing the value of x;

the difference between the third diagonal products, $ca' - c'a$, is the numerator of the fraction representing the value of y.

$$\therefore\ x = \frac{bc' - b'c}{ab' - a'b},$$

and

$$y = \frac{ca' - c'a}{ab' - a'b}.$$

531. These cross-product differences, when written in the form of square matrices, are called **Determinants.**

Thus,

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}, \quad \begin{vmatrix} b & c \\ b' & c' \end{vmatrix}, \quad \begin{vmatrix} c & a \\ c' & a' \end{vmatrix}.$$

The values of x and y may be written in the form of a determinant, as follows:

$$x = \frac{\begin{vmatrix} b & c \\ b' & c' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} c & a \\ c' & a' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}.$$

Model Solution

Solve $\begin{Bmatrix} 3x - 7y = -5, \\ 5x + 3y = 21, \end{Bmatrix}$ by the method of determinants.

$$\begin{matrix} 3 & -7 & +5, \\ 5 & +3 & -21. \end{matrix}$$

$$x = \frac{\begin{vmatrix} -7 & +5 \\ +3 & -21 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 5 & +3 \end{vmatrix}} = \frac{147 - 15}{9 + 35} = \frac{132}{44} = 3,$$

$$y = \frac{\begin{vmatrix} +5 & +3 \\ -21 & +5 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 5 & +3 \end{vmatrix}} = \frac{25 + 63}{9 + 35} = \frac{88}{44} = 2.$$

532. The quantities a, b', a', b, in the determinant $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$ are called the **Elements** of the determinant; the diagonal products, ab' and $a'b$, are called its **Terms**; the horizontal line of elements, $a\ b$, is called a **Row**; the vertical line $\begin{vmatrix} a \\ a' \end{vmatrix}$ is called a **Column.**

533. The **Order** of a determinant is the number of elements in a row or column. $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$ is a determinant of the 2d order.

534. $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$ is the **Unexpanded Form** of the determinant; $ab' - a'b$ is the **Expanded Form**; ab' is called the **Principal Diagonal**, and $a'b$ the **Secondary Diagonal.**

EXAMPLES

Expand:

1. $\begin{vmatrix} 3 & 5 \\ 7 & 8 \end{vmatrix}$ 2. $\begin{vmatrix} -7 & -8 \\ +6 & +9 \end{vmatrix}$ 3. $\begin{vmatrix} 0 & 10 \\ 5 & 20 \end{vmatrix}$ 4. $\begin{vmatrix} 16 & -5 \\ 20 & +0 \end{vmatrix}$ 5. $\begin{vmatrix} +5 & +4 \\ -3 & +7 \end{vmatrix}$

Solve:

6. $\begin{cases} 3x - 7y = 5, \\ 2x + 5y = 17. \end{cases}$

7. $\begin{cases} 4x + 5y = -6, \\ 7x - 8y = 15. \end{cases}$

8. $\begin{cases} 3x - 2y - 7 = 0, \\ 8x + 2y = 48. \end{cases}$

9. $\begin{cases} 5x + 6y + 8 = 0, \\ 9x - 10y + 11 = 0. \end{cases}$

10. $\begin{cases} 8x - 9y - 1 = 0, \\ 6x - 3y = 4x. \end{cases}$

11. $\begin{cases} a_1x - b_1y + c_1 = 0, \\ a_2x + b_2y - c_2 = 0. \end{cases}$

12. $\begin{cases} 2\frac{1}{3}x + 3\frac{1}{4}y - 74 = 0, \\ 4\frac{1}{5}x - 5\frac{1}{6}y - 1 = 0. \end{cases}$

13. $\begin{cases} 3\frac{3}{4}x - 4\frac{4}{5}y + 6 = 0, \\ 5\frac{5}{6}x + 6\frac{6}{7}y - 8 = 0. \end{cases}$

THIRD ORDER

535. If
$$\begin{cases} (1)\ a_1x + b_1y + c_1z = 0, \\ (2)\ a_2x + b_2y + c_2z = 0, \\ (3)\ a_3x + b_3y + c_3z = 0, \end{cases}$$
then

(4) $\{a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)\}\, x = 0,$

by eliminating y and z by the method of addition. This is quickly done by multiplying (1) by $b_2c_3 - b_3c_2$, (2) by $-b_1c_3 + b_3c_1$, and (3) by $b_1c_2 - b_2c_1$, and then adding the three results.

If x in (4) $\neq 0$, then must

(5) $\quad a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) = 0.$

But
$$b_2c_3 - b_3c_2 \equiv \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},$$
$$b_1c_3 - b_3c_1 \equiv \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix},$$
$$b_1c_2 - b_2c_1 \equiv \begin{vmatrix} b_1 & c_1 \\ b_2 & b_2 \end{vmatrix}.$$

$$\therefore\ (5) \equiv a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 0.$$

$$\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

536. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a determinant of the **Third Order**.

It is sometimes written in the abbreviated form
$$|\,a_1 \quad b_2 \quad c_3\,|, \text{ or } \sum \pm a_1b_2c_3.$$

537. Prop. 1. The value of the determinant is not altered by changing the columns into rows, and the rows into columns.

Dem.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\equiv 1.\quad a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\equiv 2.\quad a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

$$\equiv 3.\quad a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$\equiv 4.\quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

$$\therefore 5.\quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

EXPANSION OF A DETERMINANT OF THE THIRD ORDER

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \text{ from (3).}$$

538. From this it may be seen that a determinant can be reduced to the algebraic sum of a_1, b_1, c_1 times a determinant of the second order. These determinants of the second order are called **Minors** of the original determinant. $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is the first minor, $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ is the second minor, etc. It is seen that the minor which is the coefficient of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$; it is composed of the elements not found in the same column and the same row in which a_1 is found. It

is seen also that the same law holds true in the case of the minor to b_1 and c_1. To preserve the order of the letters, $b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}$ is written $-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$. These two expressions are equal,

$$\because\ b_1(c_2a_3 - c_3a_2) \equiv -b_1(a_2c_3 - a_3c_2).$$

Model Solution

1. $$\begin{vmatrix} 2 & 5 & -8 \\ 3 & 6 & -9 \\ 4 & -7 & 10 \end{vmatrix} = 2\begin{vmatrix} 6 & -9 \\ -7 & 10 \end{vmatrix} - 5\begin{vmatrix} 3 & -9 \\ 4 & 10 \end{vmatrix} - 8\begin{vmatrix} 3 & 6 \\ 4 & -7 \end{vmatrix}$$

$$= 2(60-63) - 5(30+36) - 8(-21-24)$$

$$= -6 - 330 + 360$$

$$= +24.$$

2. By diagram No. 1.

The terms composed of the elements of the principal diagonal and of the minor diagonals parallel to it are +, while those formed of the elements of the secondary diagonal and of the minor diagonals parallel to it are −.

$$\begin{vmatrix} 3 & -6 & 9 \\ 4 & 7 & -10 \\ 5 & -8 & 11 \end{vmatrix}$$

$$=(3\cdot 7\cdot 11) + (4\cdot -8\cdot 9)$$
$$+(5\cdot -10\cdot -6) - (5\cdot 7\cdot 9)$$
$$-(4\cdot -6\cdot 11) - (3\cdot -10\cdot -8) = -48.$$

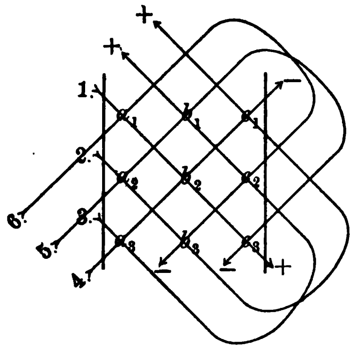


Diagram No. 1

Explanation. From the sum of the products of the elements found on lines 1, 2, 3, in diagram No. 1, subtract the product of the elements found on lines 4, 5, 6.

3. By diagram No. 2.

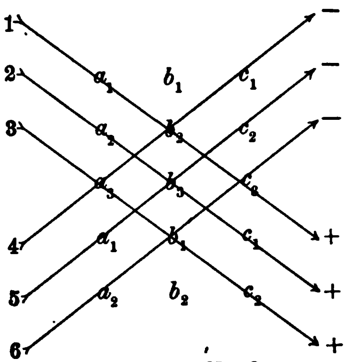


DIAGRAM NO. 2

By repeating the first and second rows of diagram No. 1, diagram No. 2 is formed. The diagonal products 1, 2, 3 are positive, while the diagonal products 4, 5, 6 are negative.

Rule. *From the sum of the diagonal products* 1, 2, 3, *subtract the diagonal products* 4, 5, 6.

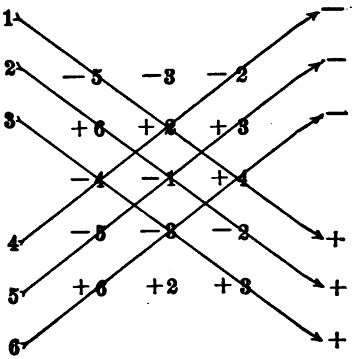
ILLUSTRATION.

$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$

Expand $\begin{vmatrix} -5 & -3 & -2 \\ +6 & +2 & +3 \\ -4 & -1 & +4 \end{vmatrix}$

Expansion

$$(-40)+(12)+(36)-(16)$$
$$-(15)-(-72)=49.$$



After a little practice the student will not need to draw the arrows, or even to repeat the first and second rows of figures.

539. Prop. 2. Interchanging two adjacent columns or rows of the determinant changes its sign but not its arithmetical value.

Dem.

1. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$

2. $$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = b_1a_2c_3 + b_2a_3c_1 + b_3a_1c_2 - b_3a_2c_1 - b_1a_3c_2 - b_2a_1c_3$$
$$= -(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3).$$

$\therefore$ (1) $\equiv$ (2) arithmetically, but with opposite signs.

540. Prop. 3. Multiplying each element in any column, or in any row, by the same number multiplies the determinant by that number.

Dem.
$$\begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} = x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

For $a_1b_2c_3x + a_2b_3c_1x + a_3b_1c_2x - a_3b_2c_1x - a_1b_3c_2x - a_2b_1c_3x$
$$\equiv x(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3).$$

541. Prop. 4. When two columns or two rows of a determinant are identical, the value of the determinant is zero.

Dem. Let
$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = v.$$

Then
$$\begin{vmatrix} a_1 & c_1 & a_1 \\ a_2 & c_2 & a_2 \\ a_3 & c_3 & a_3 \end{vmatrix} = -v, \text{ by Prop. 2.}$$

$$\therefore\ v = -v.$$

But $\qquad v \neq -v$ unless $v = 0$.

$$\therefore \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0.$$

542. Cor. When each element of one column or row is the same multiple of the corresponding element of another column or row, the value of the determinant is zero.

Illustration. $\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 2 \\ 8 & 4 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 & 1 \\ 3 & 3 & 2 \\ 4 & 4 & 3 \end{vmatrix} = 2 \times 0 = 0.$

543. Prop. 5. When each element in any column or row consists of two terms, the determinant is equivalent to the sum of two other determinants.

Dem. $\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$

Let the student demonstrate the proposition by expansion.

Prove that

$$\begin{vmatrix} a_1+x_1 & b_1+y_1 & c_1 \\ a_2+x_2 & b_2+y_2 & c_2 \\ a_3+x_3 & b_3+y_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & y_1 & c_1 \\ a_2 & y_2 & c_2 \\ a_3 & y_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix}$$

Model Solution

$$\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix} \equiv \begin{vmatrix} 13 & 13+3 & 13+6 \\ 14 & 14+3 & 14+6 \\ 15 & 15+3 & 15+6 \end{vmatrix}$$

$$\equiv \begin{vmatrix} 13 & 13 & 13 \\ 14 & 14 & 14 \\ 15 & 15 & 15 \end{vmatrix} + \begin{vmatrix} 13 & 13 & 6 \\ 14 & 14 & 6 \\ 15 & 15 & 6 \end{vmatrix} + \begin{vmatrix} 13 & 3 & 6 \\ 14 & 3 & 6 \\ 15 & 3 & 6 \end{vmatrix} + \begin{vmatrix} 13 & 3 & 13 \\ 14 & 3 & 14 \\ 15 & 3 & 14 \end{vmatrix}$$

$$\equiv 0 + 0 + 0 + 0$$

$$\equiv 0.$$

544. Prop. 6. A determinant can be reduced by replacing any one of the columns or rows by a new column or row formed by adding to, or subtracting from, the elements of the column or row to be replaced, any equimultiples of the corresponding elements of one or more of the other columns or rows, care being taken to leave one column or row unaltered.

Dem.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1+xb_1-yc_1 & b_1 & c_1 \\ a_2+xb_2-yc_2 & b_2 & c_2 \\ a_3+xb_3-yc_3 & b_3 & c_3 \end{vmatrix}$$

For
$$\begin{vmatrix} a_1+xb_1-yc_1 & b_1 & c_1 \\ a_2+xb_2-yc_2 & b_2 & c_2 \\ a_3+xb_3+yc_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} xb_1 & b_1 & c_1 \\ xb_2 & b_2 & c_2 \\ xb_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} yb_1 & b_1 & c_1 \\ yb_2 & b_2 & c_2 \\ yb_3 & b_3 & c_3 \end{vmatrix}$$

$$\equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0 - 0.$$

Model Solution

$$\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix} \equiv \begin{vmatrix} 13 & 16 & 3 \\ 14 & 17 & 3 \\ 15 & 18 & 3 \end{vmatrix} \equiv \begin{vmatrix} 13 & 3 & 3 \\ 14 & 3 & 3 \\ 15 & 3 & 3 \end{vmatrix} = 0.$$

545. Problem 1. **To solve a system of three simultaneous linear equations.**

Let the following equations be such a system:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Solving by the method of addition,

$$x = \frac{b_1c_2d_3 + b_2c_3d_1 + b_3c_1d_2 - b_3c_2d_1 - b_1c_3d_2 - b_2c_1d_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}$$

$$\equiv \frac{|b_1 \quad c_2 \quad d_3|}{|a_1 \quad b_2 \quad c_3|} \equiv \frac{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$y = \frac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}$$

$$\equiv \frac{|a_1 \quad d_2 \quad c_3|}{|a_1 \quad b_2 \quad c_3|} \equiv \frac{-|c_1 \quad d_2 \quad a_3|}{|a_1 \quad b_2 \quad c_3|} \equiv \frac{-\begin{vmatrix} c_1 & d_1 & a_1 \\ c_2 & d_2 & a_2 \\ c_3 & d_3 & a_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$z = \frac{d_1a_2b_3 + d_2a_3b_1 + d_3a_1b_2 - d_3a_2b_1 - d_1a_3b_2 - d_2a_1b_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}$$

$$\equiv \frac{|d_1 \quad a_2 \quad b_3|}{|a_1 \quad b_2 \quad c_3|} \equiv \frac{\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

MODEL SOLUTIONS

Solve $\begin{cases} x - 2y + 3z = 6, \\ 2x + 3y - 4z = 20, \\ 3x - 2y + 5z = 26. \end{cases}$

1.

$$x = \frac{\begin{vmatrix} -2 & +3 & +6 \\ +3 & -4 & +20 \\ -2 & +5 & +26 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & +3 \\ 2 & +3 & -4 \\ 3 & -2 & +5 \end{vmatrix}} = \frac{208 + 90 - 120 - 48 + 200 - 234}{15 - 12 + 24 - 27 - 8 + 20} = \frac{96}{12} = 8.$$

$$y = \frac{-\begin{vmatrix} 3 & 6 & 1 \\ -4 & 20 & 2 \\ +5 & 26 & 3 \end{vmatrix}}{12} = \frac{-(180 - 104 + 60 - 100 - 156 + 72)}{12} = \frac{48}{12} = 4.$$

$$z = \frac{\begin{vmatrix} 6 & 1 & -2 \\ 20 & 2 & +3 \\ 26 & 3 & -2 \end{vmatrix}}{12} = \frac{-24 - 120 + 78 + 104 - 54 + 40}{12} = \frac{24}{12} = 2.$$

2.

1. $$\begin{vmatrix} 1 & -2 & +3 \\ 2 & +3 & -4 \\ 3 & -2 & +5 \end{vmatrix} \equiv \begin{vmatrix} 1 & 0 & 0 \\ 2 & 7 & -10 \\ 3 & 4 & -4 \end{vmatrix} \equiv 1 \begin{vmatrix} 7 & -10 \\ 4 & -4 \end{vmatrix} \equiv -28 + 40 = 12.$$

2. $$\begin{vmatrix} -2 & +3 & 6 \\ +3 & -4 & 20 \\ -2 & +5 & 26 \end{vmatrix} \equiv \begin{vmatrix} -6 & +6 & 6 \\ +9 & -8 & 20 \\ -6 & +10 & 36 \end{vmatrix} \div 6 \equiv \begin{vmatrix} -6 & 0 & 0 \\ +9 & 1 & 29 \\ -6 & 4 & 20 \end{vmatrix} \div 6$$

$$\equiv \begin{vmatrix} -1 & 0 & 0 \\ +\frac{3}{2} & 1 & 29 \\ -1 & 4 & 20 \end{vmatrix} \equiv -1 \begin{vmatrix} 1 & 29 \\ 4 & 20 \end{vmatrix} = 96.$$

3. $$-\begin{vmatrix} 3 & 6 & 1 \\ -4 & 20 & 2 \\ 5 & 26 & 3 \end{vmatrix} \equiv -\begin{vmatrix} 0 & 0 & 1 \\ -10 & 8 & 2 \\ -4 & 8 & 3 \end{vmatrix} \equiv -1 \begin{vmatrix} -10 & 8 \\ -4 & 8 \end{vmatrix} = 48.$$

4. $$\begin{vmatrix} 6 & 1 & -2 \\ 20 & 2 & +3 \\ 26 & 3 & -2 \end{vmatrix} \equiv \begin{vmatrix} 0 & 1 & 0 \\ 8 & 2 & 7 \\ 8 & 3 & 4 \end{vmatrix} \equiv 1 \begin{vmatrix} 7 & 8 \\ 4 & 8 \end{vmatrix} = 24.$$

$$\therefore\ x = \tfrac{96}{12} = 8; \quad y = \tfrac{48}{12} = 4; \quad z = \tfrac{24}{12} = 2.$$

EXAMPLES

Find the values of the following determinants:

1. $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & a^3 & a^2 \\ a^3 & 1 & a \\ a^2 & a & 1 \end{vmatrix}$ 3. $\begin{vmatrix} a^2 & a & 1 \\ 1 & a^2 & a \\ a & 1 & a^2 \end{vmatrix}$

4. $\begin{vmatrix} 3 & 13 & 23 \\ 7 & 30 & 53 \\ 9 & 39 & 70 \end{vmatrix}$ 7. $\begin{vmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{vmatrix}$

5. $\begin{vmatrix} 1 & -4 & 7 \\ 2 & -5 & -8 \\ 3 & -6 & -9 \end{vmatrix}$ 8. $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$

6 $\begin{vmatrix} 1 & -7 & -13 \\ 2 & 10 & 15 \\ 3 & 12 & -16 \end{vmatrix}$ 9. $\begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix}$

Solve for x:

10. $\begin{vmatrix} 1 & 1 & x \\ 2 & 2 & 2 \\ 3 & x & 3 \end{vmatrix} = 0.$ 12. $\dfrac{1}{x} = \dfrac{\begin{vmatrix} 2 & -4 \\ 10 & 8 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 6 & -9 \end{vmatrix}}$

11. $\begin{vmatrix} x & a & a \\ b & b & b \\ c & x & c \end{vmatrix} = 0.$ 13. $\begin{vmatrix} 5 & x & 5 \\ 6 & 7 & x \\ 7 & 8 & 7 \end{vmatrix} = 0.$

Prove:

14. $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix} \equiv 132.$ 15. $\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} \equiv -360.$

16. $\begin{vmatrix} 1 & a & -b \\ -a & 1 & c \\ b & -c & 1 \end{vmatrix} \equiv 1 + a^2 + b^2 + c^2.$

17. $$\begin{vmatrix} g+h & f-h & f-g \\ g-h & h+f & g-f \\ h-g & h-f & f+g \end{vmatrix} \equiv 8fgh.$$

18. $$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \equiv (x-y)(y-z)(z-x).$$

19. $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} \equiv ab+ac+bc+abc.$$

20. $$\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & -x+y-z & 2y \\ 2z & 2z & -x-y+z \end{vmatrix} \equiv (x+y+z)^3.$$

Solve the following equations by determinants, using any one of the preceding methods, and check the results, using another method. Also verify the roots by substituting in each equation:

1. $$\begin{cases} 2x-3y=3, \\ 3y-4z=7, \\ 4z-5x=2. \end{cases}$$

2. $$\begin{cases} 7x-3y=30, \\ 9y-5z=34, \\ x+y+z=33. \end{cases}$$

3. $$\begin{cases} 3x-y+z=17, \\ 7x+4y-5z=3, \\ 5x+3y-2z=10. \end{cases}$$

4. $$\begin{cases} 4x-5y+2z=6, \\ 2x+3y-z=20, \\ 7x-4y+3z=35. \end{cases}$$

5. $$\begin{cases} bz+cy=a, \\ az+cx=b, \\ ay+bx=c. \end{cases}$$

6. $$\begin{cases} 2x+5y+3z=13, \\ 2x+2y-z=12, \\ 5x+5y-2z=29. \end{cases}$$

7. $$\begin{cases} 2x+3y+4z=20, \\ 3x+4y+5z=26, \\ 4x-3y-2z=-8. \end{cases}$$

8. $$\begin{cases} x+3y-7z=31, \\ 3x+y+5z=-49, \\ 20x+2y-5z=-35. \end{cases}$$

9. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=3, \\ \dfrac{a}{x}-\dfrac{b}{y}+\dfrac{c}{z}=1, \\ \dfrac{2a}{x}-\dfrac{b}{y}-\dfrac{c}{z}=0. \end{cases}$

11. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}=a, \\ \dfrac{1}{x}-\dfrac{1}{y}+\dfrac{1}{z}=b, \\ \dfrac{1}{y}+\dfrac{1}{z}-\dfrac{1}{x}=c. \end{cases}$

10. $\begin{cases} x+2y+2z=11, \\ 2x+y+z=7, \\ 3x+4y+z=14. \end{cases}$

12. $\begin{cases} 2x-3y+5z=1, \\ 3x-y+2z=3, \\ 4x+5y-7z=7. \end{cases}$

FOURTH AND HIGHER ORDERS

546. $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ is a determinant of the **Fourth Order.**

A determinant of the fourth order can be expressed as the sum of four determinants of the third order, and then expanded in the usual manner.

Model Solutions

1. $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

$$\equiv a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix},$$

or, $$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} c_2 & d_2 & a_2 \\ c_3 & d_3 & a_3 \\ c_4 & d_4 & a_4 \end{vmatrix} + c_1\begin{vmatrix} d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \\ d_4 & a_4 & b_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

2. $\begin{vmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{vmatrix}$

$$\equiv 1\begin{vmatrix} 6 & 10 & 14 \\ 7 & 11 & 15 \\ 8 & 12 & 16 \end{vmatrix} - 5\begin{vmatrix} 10 & 14 & 2 \\ 11 & 15 & 3 \\ 12 & 16 & 4 \end{vmatrix} + 9\begin{vmatrix} 13 & 2 & 6 \\ 14 & 3 & 7 \\ 15 & 4 & 8 \end{vmatrix} - 13\begin{vmatrix} 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{vmatrix}$$

$$\equiv (3432 - 3432) - 5(1456 - 1456) + 9(858 - 858)$$
$$- 13(672 - 672) = 0.$$

3. $\begin{vmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{vmatrix} \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -4 & -8 & -12 \\ 3 & -8 & -16 & -24 \\ 5 & -12 & -24 & -36 \end{vmatrix} \equiv -1\begin{vmatrix} 4 & 8 & 12 \\ 8 & 16 & 24 \\ 12 & 24 & 36 \end{vmatrix}$

$= 0$ by Cor. under Prop. 4.

547. Problem 2. **To solve the system**

$$\begin{cases} a_1x + b_1y + c_1z + d_1v = k_1, \\ a_2x + b_2y + c_2z + d_2v = k_2, \\ a_3x + b_3y + c_3z + d_3v = k_3, \\ a_4x + b_4y + c_4z + d_4v = k_4. \end{cases}$$

Solution

$$x = \frac{|k_1b_2c_3d_4|}{|a_1b_2c_3d_4|}, \qquad z = \frac{|a_1b_2k_3d_4|}{|a_1b_2c_3d_4|},$$

$$y = \frac{|a_1k_2c_3d_4|}{|a_1b_2c_3d_4|}, \qquad v = \frac{|a_1b_2c_3k_4|}{|a_1b_2c_3d_4|}.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ a_3 & b_3 & c_3 & \cdots & l_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix}$$

is a determinant of the *nth order*, and can be reduced to n determinants of the $(n-1)$th order. Its abbreviation is $|a_1 \ b_2 \ c_3 \ \ldots \ l_n|$, or $\Sigma \pm a_1 b_2 c_3 \ldots l_n$.

MISCELLANEOUS EXAMPLES

Find the value of :

1. $\begin{vmatrix} a & b & a \\ c & c & b \\ b & a & c \end{vmatrix}$

2. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 9 \\ 1 & 4 & 10 & 15 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{vmatrix}$.

4. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix}$

5. $\begin{vmatrix} 7 & 13 & 10 & 6 \\ 5 & 9 & 7 & 4 \\ 8 & 12 & 11 & 7 \\ 4 & 10 & 6 & 3 \end{vmatrix}$

6. $\begin{vmatrix} 3 & 8 & 4 \\ -7 & 9 & 5 \\ +4 & -13 & 6 \end{vmatrix}$

Prove that

7. $\begin{vmatrix} 9 & 13 & 17 & 4 \\ 8 & 28 & 33 & 8 \\ 30 & 40 & 54 & 13 \\ 24 & 37 & 46 & 11 \end{vmatrix} = -15.$

8. $\begin{vmatrix} 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 2 & -3 & 4 \\ 1 & 2 & 5 & 2 & -1 \\ 2 & -3 & 2 & 1 & 0 \\ -5 & 4 & -1 & 0 & -15 \end{vmatrix} = 0.$

9. $\begin{vmatrix} 30 & 11 & 20 & 38 \\ 6 & 3 & 0 & 9 \\ 11 & -2 & 36 & 3 \\ 19 & 6 & 17 & 22 \end{vmatrix} = 9.$

10. $\begin{vmatrix} 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & 2 \\ 0 & -2 & 5 & 3 \\ 5 & 2 & 3 & 4 \end{vmatrix} = -972.$

Solve :

11. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1, \\ \dfrac{x}{a}+\dfrac{y}{c}+\dfrac{z}{b}=1, \\ \dfrac{x}{b}+\dfrac{y}{a}+\dfrac{z}{c}=1. \end{cases}$

12. $\begin{cases} 7x-3y=1, \\ 11z-7u=1, \\ 4z-7y=1, \\ 19x-3u=1. \end{cases}$

13. $\begin{vmatrix} 3 & 4 & 5 \\ 2x & 3x & 3 \\ x^3 & x^2 & 2 \end{vmatrix}=0.$

14. $\begin{vmatrix} x-a & x-b & x-c \\ x-b & x-c & x-a \\ x-c & x-a & x-b \end{vmatrix}=0.$

15. $\begin{cases} x+y=\frac{5}{6}xy, \\ x+z=\frac{3}{10}xz, \\ y+z=\frac{2}{15}yz. \end{cases}$

16. $\begin{cases} 7x-2z+3u=17, \\ 4y-2z+v=11, \\ 5y-3x-2u=8, \\ 4y-3u+2v=9, \\ 3z+8u=33. \end{cases}$

17. $\begin{cases} ax+by+cz=A, \\ a^2x+b^2y+c^2z=A^2, \\ a^3x+b^3y+c^3z=A^3. \end{cases}$

18. $\begin{cases} x+y+z=a+b+c, \\ bx+cy+az=a^2+b^2+c^2, \\ cx+ay+bz=a^2+b^2+c^2. \end{cases}$

19. $xyz=a(yz-xz-xy)=b(zx-xy-yz)=c(xy-yz-zx).$